Video 4: Big-O Notation

COMS10017 - (Object-Oriented Programming and) Algorithms

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Big O Notation

Definition: O-notation ("Big O")

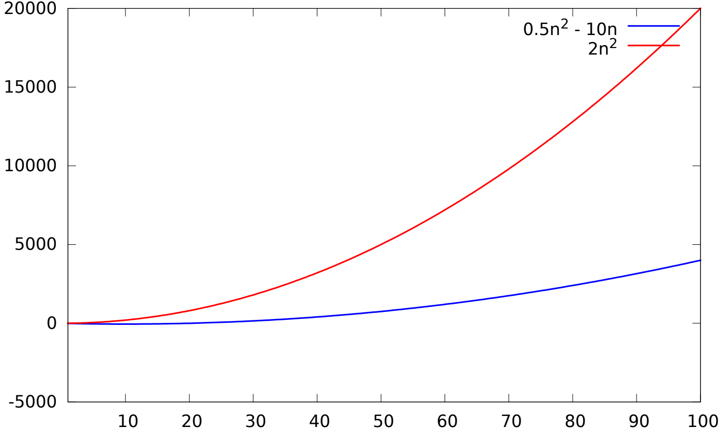
Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be a function. Then $O(g(n))$ is the set of functions:

$$O(g(n)) = \{f(n) : \text{There exist positive constants } c \text{ and } n_0 \text{ such that } 0 \leq f(n) \leq cg(n) \text{ for all } n \geq n_0\}$$

Meaning: $f(n) \in O(g(n))$: "g grows asymptotically at least as fast as f up to constants"

O-Notation: Example

Example: $f(n) = \frac{1}{2}n^2 - 10n$ and $g(n) = 2n^2$



Then: $g(n) \in O(f(n))$, since $6f(n) \geq g(n)$, for every $n \geq n_0 = 60$

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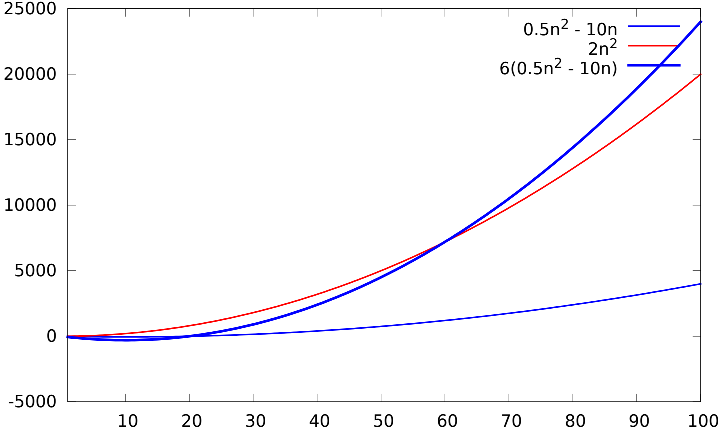

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More Examples/Exercises

Recall:

$$O(g(n)) = \{f(n) : \text{There exist positive constants } c \text{ and } n_0 \text{ such that } 0 \leq f(n) \leq cg(n) \text{ for all } n \geq n_0\}$$

Exercises:

- $100n \overset{?}{\in} O(n)$ Yes, chose $c = 100, n_0 = 1$
- $0.5n \overset{?}{\in} O(n/\log n)$ No: Suppose that such constants c and n_0 exist. Then, for every $n \geq n_0$:

$$\begin{aligned} 0.5n &\leq cn/\log n \\ \log n &\leq 2c \\ n &\leq 2^{2c} \text{ , a contradiction,} \end{aligned}$$

since this does not hold for every $n > 2^{2c}$.

Proving that $f \in O(g)$:

Find constants c, n_0 as in the statement of the definition of Big-O, i.e., such that $f(n) \leq c \cdot g(n)$, for all $n \geq n_0$

Proving that $f \notin O(g)$:

Proof by contradiction: Assume that constants c, n_0 exist as in the statement of the definition of Big-O and derive a contradiction

Sum of Two Functions

Lemma (Sum of Two Functions)

Suppose that $f, g \in O(h)$. Then: $f + g \in O(h)$.

Proof.

To Do: We need to find constants C, N_0 such that

$$f(n) + g(n) \leq C \cdot h(n), \text{ for every } n \geq N_0 \ .$$

Since $f \in O(h)$ there exist constants c, n_0 such that

$$f(n) \leq c \cdot h(n), \text{ for every } n \geq n_0 \ .$$

Since $g \in O(h)$ there exist constants c', n_0' such that

$$g(n) \leq c' h(n), \text{ for every } n \geq n_0' \ .$$

Let $C = c + c'$ and let $N_0 = \max\{n_0, n_0'\}$. Then:

$$f(n) + g(n) \leq ch(n) + c'h(n) = C \cdot h(n) \text{ for every } n \geq N_0 \ . \quad \square$$

Further Properties

Lemma (Polynomials)

Let $f(n) = c_0 + c_1 n + c_2 n^2 + c_3 n^3 + \cdots + c_k n^k$, for some integer k that is independent of n. Then: $f(n) \in O(n^k)$.

Proof: Apply statement on last slide $O(1)$ times (k times) $\square$

Attention: Wrong proof of $n^2 \in O(n)$: (this is clearly wrong)

$$\begin{aligned}
n^2 &= n + n + \underbrace{n + \ldots n}_{n-2 \text{ times}} = O(n) + O(n) + \underbrace{n + \ldots n}_{n-2 \text{ times}} \\
&= O(n) + \underbrace{n + \ldots n}_{n-2 \text{ times}} = O(n) + O(n) + \underbrace{n + \ldots n}_{n-3 \text{ times}} = \\
&= O(n) + \underbrace{n + \ldots n}_{n-3 \text{ times}} = \cdots = O(n) \ .
\end{aligned}$$

Application of statement on last slide n times! (only allowed to apply statement $O(1)$ times!)

Runtime of Algorithms

Tool for the Analysis of Algorithms

- We will express the runtime of algorithms using O-notation
- This allows us to compare the runtimes of algorithms
- **Important:** Find the slowest growing function f such that our runtime is in $O(f)$ (most algorithms have a runtime of $O(2^n)$)

Important Properties for the Analysis of Algorithms

- Composition of instructions:

$$f \in O(h_1), g \in O(h_2) \text{ then } f + g \in O(h_1 + h_2)$$

- Loops: (repetition of instructions)

$$f \in O(h_1), g \in O(h_2) \text{ then } f \cdot g \in O(h_1 \cdot h_2)$$

Hierachy

Rough incomplete Hierachy

- Constant time: $O(1)$ (individual operations)
- Sub-logarithmic time: e.g., $O(\log \log n)$
- Logarithmic time: $O(\log n)$ (FAST-PEAK-FINDING)
- Poly-logarithmic time: e.g., $O(\log^2 n), O(\log^{10} n), \ldots$
- Linear time: $O(n)$ (e.g., time to read the input)
- Quadratic time: $O(n^2)$ (potentially slow on big inputs)
- Polynomial time: $O(n^c)$ (used to be considered efficient)
- Exponential time: $O(2^n)$ (works only on very small inputs)
- Super-exponential time: e.g. $O(2^{2^n})$ (big trouble...)