Video 13: Heap Sort

COMS10017 - (Object-Oriented Programming and) Algorithms

Dr Christian Konrad

Sorting Algorithms seen so far

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- Insertion-Sort: $O(n^2)$ in worst, in place, stable
- Merge-Sort: $O(n \log n)$ in worst case, NOT in place, stable

Heap Sort (best of the two)

- $O(n \log n)$ in worst case, in place, **NOT** stable
- Uses a *heap data structure* (a heap is special tree)

Data Structures

- *Data storage format that allows for efficient access and modification*
- Building block of many efficient algorithms
- For example, an array is a data structure

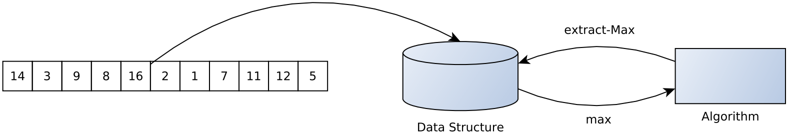
Priority Queues

Priority Queue:

Data structure that allows the following operations:

- Build(.): Create data structure given a set of data items
- Extract-Max(.): Remove the maximum element from the data structure and return it
- *others...*

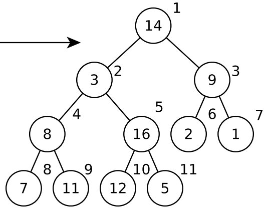
Sorting using a Priority Queue



From Array to Tree

Interpretation of an Array as a Complete Binary Tree

1	2	3	4	5	6	7	8	9	10	11
14	3	9	8	16	2	1	7	11	12	5



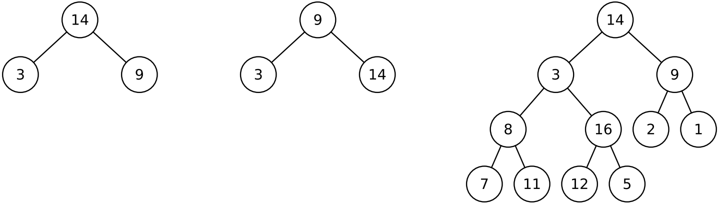
Easy Navigation:

- Parent of i: $\lfloor i/2 \rfloor$
- Left/Right Child of i: $2i$ and $2i + 1$

Heap Property

The Heap Property

Key of nodes larger than keys of their children



Heap Property $\rightarrow$ Maximum at root
Important for Extract-Max(.)

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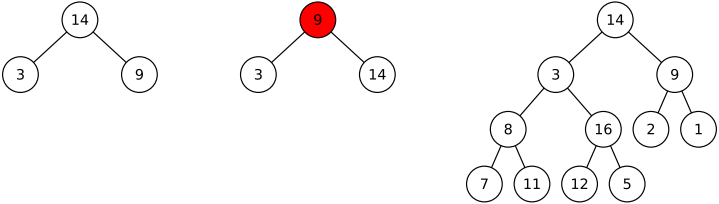

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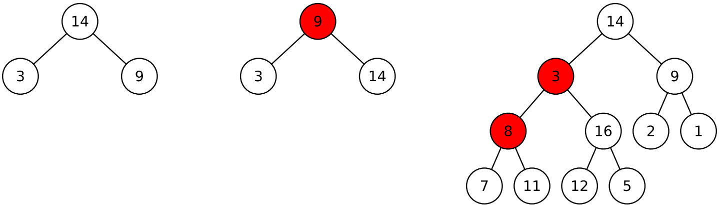

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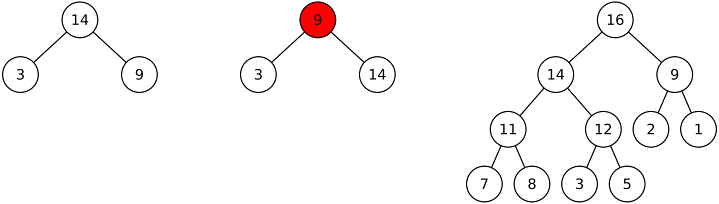

Heap Property $\rightarrow$ Maximum at root
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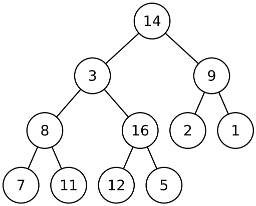
The Heapify Operation

Constructing a Heap: Build(.)

Given a binary tree, transform it into one that fulfills the Heap Property

1. Traverse tree with regards to right-to-left array ordering
2. If node does not fulfill Heap Property: **Heapify()**



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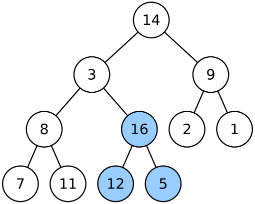

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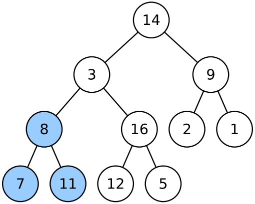

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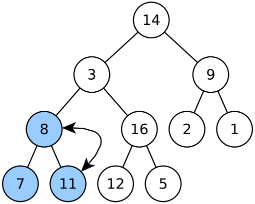

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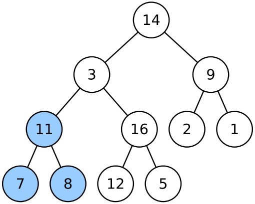

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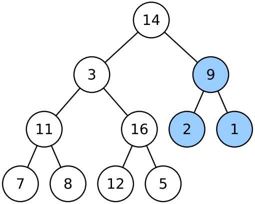

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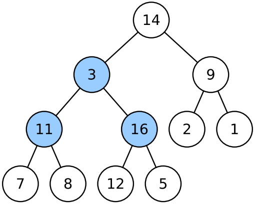

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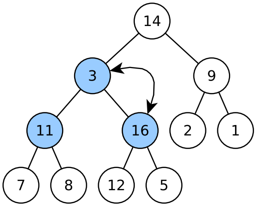

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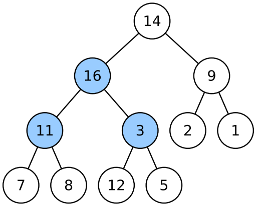

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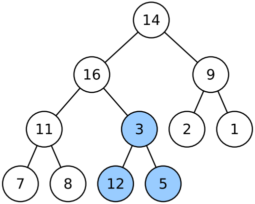

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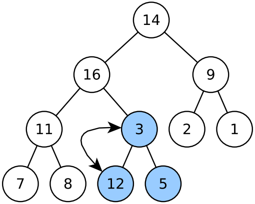

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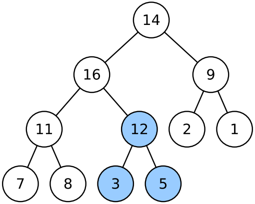

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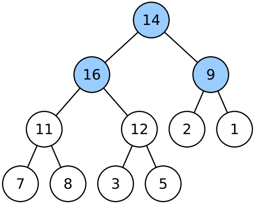

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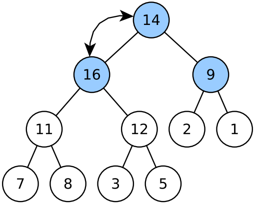

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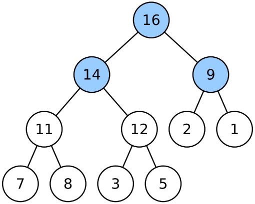


Runtime of Heapify()

Heapify()

Let p be the key of a node and let c_1, c_2 be the keys of its children

- Let $c = \max\{c_1, c_2\}$
- If $c > p$ then exchange nodes with keys p and c
- call **Heapify()** at node with key c

Runtime:

- Exchanging nodes requires time $O(1)$
- The number of recursive calls is bounded by the height of the tree, i.e., $O(\log n)$
- Runtime of **Heapify**: $O(\log n)$.

Constructing a Heap: Build(.) Runtime $O(n \log n)$

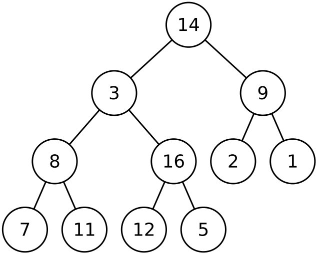
Improved Analysis of Heap Construction

More Precise Analysis of the Heap Construction Step

- Heapify(x): $O(\text{depth of subtree rooted at } x) = O(\log n)$
- **Observe:** Most nodes close to the "bottom" in a complete binary tree

Analysis:

- Let i be the largest integer such that $n' := 2^i - 1$ and $n' < n$
- There are at most n' internal nodes (candidates for Heapify())
- These nodes are contained in a perfect binary tree
- This tree has i levels



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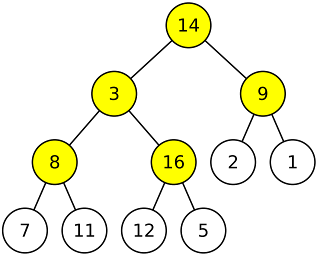

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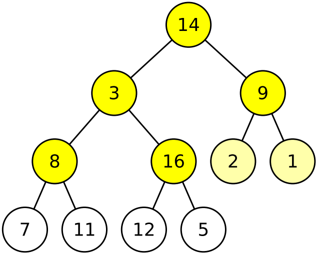
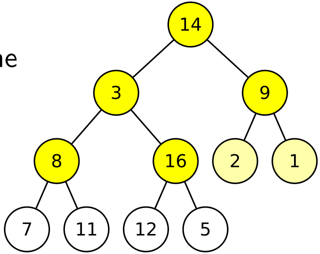


Analysis

We sum over all relevant levels, count the number of nodes per level, and multiply with the depth of their subtrees:



$$\begin{aligned}\text{Runtime} &= \sum_{j=1}^{i} \#\text{ nodes at level } (i-j+1) \cdot \text{depth of subtree} \cdot O(1) \\ &= O(1) \sum_{j=1}^{i} 2^{i-j} \cdot j = O(1) \cdot 2^i \cdot \sum_{j=1}^{i} \frac{j}{2^j} \\ &= O(2^i) = O(n') = O(n) \ ,\end{aligned}$$

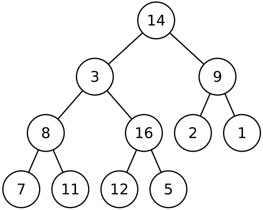
using $\sum_{j=1}^{i} \frac{j}{2^j} = O(1)$ (see trick on video 7).

The Complete Algorithm

Putting Everything Together

14	3	9	8	16	2	1	7	11	12	5

1. Build-heap()
2. Repeat n times:
   1. Swap root with last element
   2. Decrease size of heap by 1
   3. Heapify(root)



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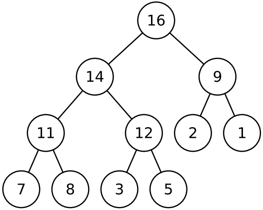
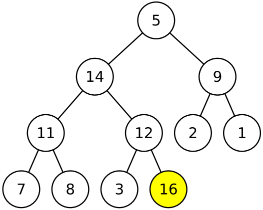


Putting Everything Together

5	14	9	11	12	2	1	7	8	3	16



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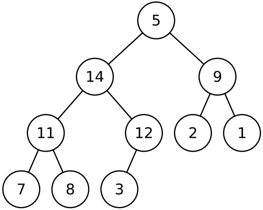

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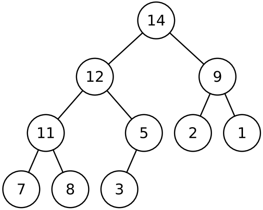

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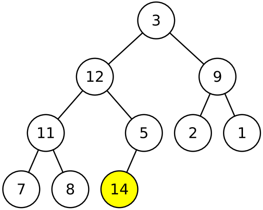

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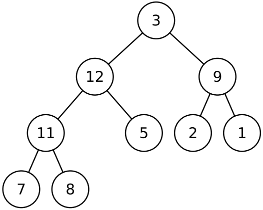

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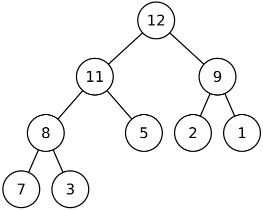

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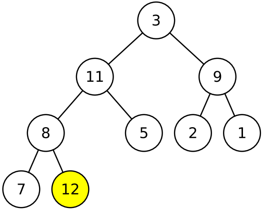

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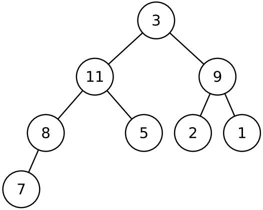

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The Complete Algorithm

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The Complete Algorithm

Putting Everything Together

1	2	3	5	7	8	9	11	12	14	16

1. Build-heap() $O(n)$
2. Repeat n times:
    1. Swap root with last element $O(1)$
    2. Decrease size of heap by 1 $O(1)$
    3. Heapify(root) $O(\log n)$

Runtime: $O(n \log n)$

Heapsort is Not Stable

Example:

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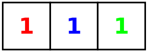

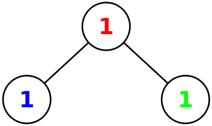

1 is moved from left to the right past 1 and 1

Heap-sort not stable

Heapsort is Not Stable

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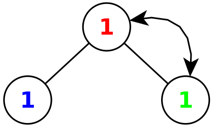
1	1	1



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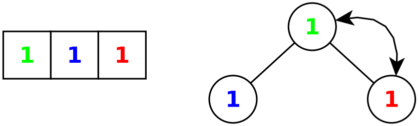

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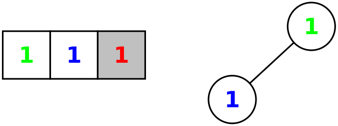

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