Video 18: Recurrences I

COMS10017 - (Object-Oriented Programming and) Algorithms

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Divide-and-conquer Algorithms

Algorithmic Design Principle: Divide-and-conquer

1. **Divide** the problem into a number of subproblems that are smaller instances of the same problem
2. **Conquer** the subproblems by solving them recursively (if subproblems have constant size, solve them *directly*)
3. **Combine** the solutions to the subproblems into the solution for the original problem

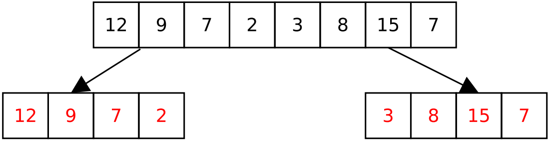
Examples

Quicksort, mergesort, maximum subarray algorithm, binary search, FAST-PEAK-FINDING, ...

Example: Merge sort

Recall: Merge Sort

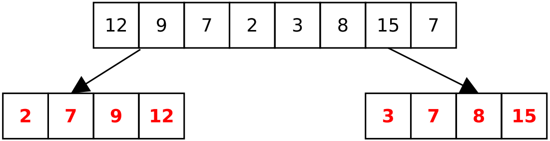
1. **Divide**
   Split input array A of length n into subarrays $A_1 = A[0, \lfloor n/2 \rfloor]$ and $A_2 = A[\lfloor n/2 \rfloor + 1, n-1]$



Example: Merge sort

Recall: Merge Sort

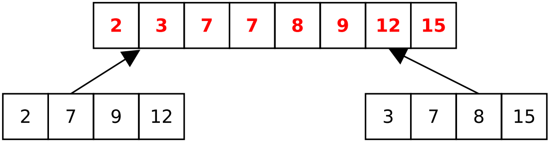
1. **Divide** $A \rightarrow A_1$ and A_2
2. **Conquer**
   Sort A_1 and A_2 recursively using the same algorithm



Example: Merge sort

Recall: Merge Sort

1. **Divide** $A \rightarrow A_1$ and A_2
2. **Conquer** Solve A_1 and A_2
3. **Combine**
   Combine sorted subarrays A_1 and A_2 and obtain sorted array A



Runtime: (assuming that n is a power of 2)

$$
\begin{aligned}
T(1) &= O(1) \\
T(n) &= 2T(n/2) + O(n)
\end{aligned}
$$

How to solve Recurrences?

Recurrences

- Divide-and-Conquer algorithms naturally lead to recurrences
- How can we *solve* them? Often only interested in asymptotic upper bounds

Methods for solving recurrences

- Substitution method
  guess solution, verify, induction
- Recursion-tree method (previously seen for merge sort and maximum subarray problem)
  may have plenty of awkward details, provides good guess that can be verified with substitution method
- Master theorem
  very powerful, cannot always be applied

The Substitution Method

1. Guess the form of the solution
2. Use mathematical induction to find the constants and show that the solution works
3. Method provides an upper bound on the recurrence

Example (suppose n is always a power of two)

$$\begin{aligned} T(1) &= O(1) \\ T(n) &= 2T(n/2) + O(n) \end{aligned}$$

The Substitution Method

1. Guess the form of the solution
2. Use mathematical induction to find the constants and show that the solution works
3. Method provides an upper bound on the recurrence

Example (suppose n is always a power of two)

$$\begin{aligned} T(1) &= c_1 \\ T(n) &= 2T(n/2) + c_2 n \end{aligned}$$

Eliminate O-notation in recurrence

Step 1. Guess good upper bound

$$T(n) \leq Cn \log n$$

The Substitution Method (2)

Step 2. Substitute into the Recurrence

- Assume that our guess $T(n) \le Cn \log n$ is correct for every $n' < n$
- Corresponds to induction step of a proof by induction

$$\begin{aligned} T(n) &= 2T(n/2) + c_2 n \ \le \ 2C\frac{n}{2}\log(\frac{n}{2}) + c_2 n \\ &= Cn\,(\log(n) - \log(2)) + c_2 n \\ &= Cn \log n - Cn + c_2 n \ \le \ Cn \log n \ , \end{aligned}$$

if we chose $C \ge c_2$. ✓

Verify the Base Case

$$T(1) \le C \cdot 1 \log(1) = 0 \not\ge c_1 \quad ✗$$

The base case is a problem...

The Substitution Method (3)

Recall: $T(1) = c_1$ and $T(n) = 2T(n/2) + c_2 n$
Our guess: $T(n) \le Cn \log n$ (induction step holds for $C \ge c_2$)

Solution: Choose a different base case! $n = 2$

$$\begin{aligned} T(2) &= 2T(1) + 2c_2 = 2c_1 + 2c_2 = 2(c_2 + c_1) \\ C2\log 2 &= 2C \end{aligned}$$

Hence, for every $C \ge c_2 + c_1$, our guess holds for $n = 2$:

$$T(2) \le C2\log 2 \ .$$

Result

- We proved $T(n) \le Cn \log n$, for every $n \ge 2$, when choosing $C \ge c_1 + c_2$
- **Observe:** This implies $T(n) \in O(n \log n)$ (important)

Asymptotic notation allows us to chose arbitrary base case

A Strange Problem

Example: Give an upper bound on the recurrence

$$\begin{aligned} T(1) &= 1 \\ T(n) &= T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + 1 \end{aligned}$$

Step 1: Guess correct solution $T(n) \le f(n) := Cn$

Step 2: Verify the solution

$$T(n) \le C\lceil n/2 \rceil + C\lfloor n/2 \rfloor + 1 = Cn + 1 \not\le f(n) \text{ ✗}$$

- We need a different guess
- Let's try: $f_1(n) := Cn + 1$ and $f_2(n) := Cn - 1$

$$\begin{aligned} f_1 : T(n) &\le C\lceil n/2 \rceil + 1 + C\lfloor n/2 \rfloor + 1 + 1 = Cn + 3 \not\le f_1(n) \text{ ✗} \\ f_2 : T(n) &\le C\lceil n/2 \rceil - 1 + C\lfloor n/2 \rfloor - 1 + 1 = Cn - 1 = f_2(n) \checkmark \end{aligned}$$

(holds for every positive C)

A Strange Problem (2)

Verify Base Case for f_2

- We have: $T(1) = 1$ and $f_2(1) = C - 1 \geq T(1)$ for $C \geq 2$
- We thus set the constant C in f_2 to $C = 2$
- Then $f_2(n) = 2n - 1 \geq T(n)$ for every $n \geq 1$
- This implies that $T(n) \in O(n)$

Comments

- Guessing right can be difficult and requires experience
- However, recursion tree method can provide a good guess!