# Video 21: Dynamic Programming - Pole Cutting

## COMS10017 - (Object-Oriented Programming and) Algorithms

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# Pole Cutting

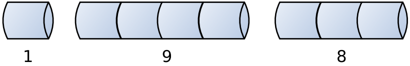
**Pole-cutting:**

- Given is a pole of length $n$

- The pole can be cut into multiple pieces of integral lengths
- A pole of length $i$ is sold for price $p(i)$, for some function $p$

**Example:**

| length $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| price $p(i)$ | 1 | 5 | 8 | 9 | 10 | 17 | 17 | 20 | 24 | 30 |

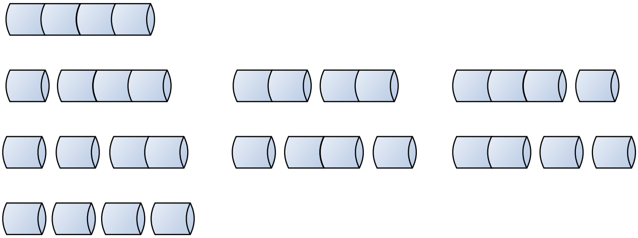


# Pole Cutting (2)

**Problem:** POLE-CUTTING

1. **Input:** Price table $p_i$, for every $i \geq 1$, length $n$ of initial pole
2. **Output:** Maximum revenue $r_n$ obtainable by cutting pole into smaller pieces

**How many ways of cutting the pole are there?**



# Pole Cutting (3)

There are $2^{n-1}$ ways to cut a pole of length $n$.

**Proof.**

There are $n-1$ positions where the pole can be cut. For each position we either cut or we don't. This gives $2^{n-1}$ possibilities. $\square$

**Problem:**

- Find best out of $2^{n-1}$ possibilities
- We could disregard similar cuts, but we would still have an exponential number of possibilities
- A fast algorithm cannot try out all possibilities

# Pole Cutting (4)

**Notation**

$$7 = 2 + 2 + 3$$

means we cut a pole of length 7 into pieces of lengths $2, 2$ and 3

**Optimal Cut**

- Suppose the optimal cut uses $k$ pieces

$$n = i_1 + i_2 + \cdots + i_k$$

- Optimal revenue $r_n$:

$$r_n = p(i_1) + p(i_2) + \cdots + p(i_k)$$

# Pole Cutting (5)

**What are the optimal revenues $r_i$?**

| length $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| price $p(i)$ | 1 | 5 | 8 | 9 | 10 | 17 | 17 | 20 | 24 | 30 |

$$
\begin{aligned}
r_1 &= 1 & 1 &= 1 \\
r_2 &= 5 & 2 &= 2 \\
r_3 &= 8 & 3 &= 3 \\
r_4 &= 10 & 4 &= 2+2 \\
r_5 &= 13 & 5 &= 2+3 \\
r_6 &= 17 & 6 &= 6 \\
r_7 &= 18 & 7 &= 2+2+3 \\
r_8 &= 22 & 8 &= 2+6 \\
r_9 &= 25 & 9 &= 3+6 \\
r_{10} &= 30 & 10 &= 10
\end{aligned}
$$

### Optimal Substructure

- Consider an optimal solution to input length $n$

$$n = i_1 + i_2 + \cdots + i_k \text{ for some } k$$

- Then:

$$n - i_1 = i_2 + \cdots + i_k$$

is an optimal solution to the problem of size $n - i_1$

### Computing Optimal Revenue $r_n$:

$$r_n = \max\{p_n, r_1 + r_{n-1}, r_2 + r_{n-2}, \ldots, r_{n-1} + r_1\}$$

- $p_n$ corresponds to the situation of no cut at all
- $r_i + r_{n-i}$: initial cut into two pieces of sizes $i$ and $n - i$

# Pole Cutting: Dynamic Programming Formulation

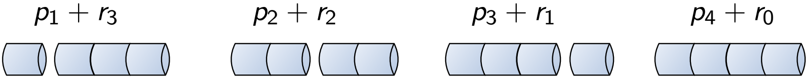
**Simpler Recursive Formulation:** Let $r_0 = 0$

$$r_n = \max_{1 \le i \le n} \left( p_i + r_{n-i} \right) .$$

**Observe:** Only one subproblem in this formulation

**Example:** $n = 4$

$$r_n = \max\{p_1 + r_3, p_2 + r_2, p_3 + r_1, p_4 + r_0\}$$



# Recursive Top-down Implementation

**Recall:**

$$r_n = \max_{1 \le i \le n} (p_i + r_{n-i}) \text{ and } r_0 = 0 \ .$$

**Direct Implementation:**

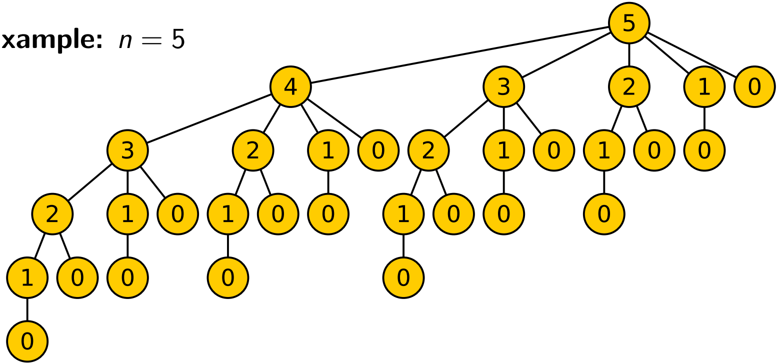
**Require:** Integer $n$, Array $p$ of length $n$ with prices
  **if** $n = 0$ **then**
    **return** 0
  $q \leftarrow -\infty$
  **for** $i = 1 \dots n$ **do**
    $q \leftarrow \max\{q, p[i] + \textsc{Cut-Pole}(p, n-i)\}$
  **return** $q$

Algorithm $\textsc{Cut-Pole}(p, n)$

**How efficient is this algorithm?**

# Recursion Tree for CUT-POLE

**Example:** $n = 5$




**Number Recursive Calls:** $T(n)$

$$T(n) = 1 + \sum_{j=0}^{n-1} T(j) \text{ and } T(0) = 1$$

# Solving Recurrence

**How to Solve this Recurrence?**

$$T(n) = 1 + \sum_{j=0}^{n-1} T(j) \text{ and } T(0) = 1$$

- Substitution Method: Using guess $T(n) = O(c^n)$, for some $c$
- **Trick:** compute $T(n) - T(n-1)$

$$\begin{aligned} T(n) - T(n-1) &= 1 + \sum_{j=0}^{n-1} T(j) - \left(1 + \sum_{j=0}^{n-2} T(j)\right) \\ &= T(n-1) \text{ , hence:} \\ T(n) &= 2T(n-1) \ . \end{aligned}$$

This implies $T(i) = 2^i$.

# Discussion

**Runtime of Cut-Pole**

- Recursion tree has $2^n$ nodes
- Each function call takes time $O(n)$ (for-loop)
- Runtime of CUT-POLE is therefore $O(n2^n)$. ($O(2^n)$ can also be argued)

**What can we do better?**

- Observe: We compute solutions to subproblems many times
- Avoid this by storing solutions to subproblems in a table!
- This is a key feature of dynamic programming

# Implementing the Dynamic Programming Approach

**Top-down** with memoization

- When computing $r_i$, store $r_i$ in a table $T$ (of size $n$)
- Before computing $r_i$ again, check in $T$ whether $r_i$ has previously been computed

**Bottom-up**

- Fill table $T$ from smallest to largest index
- No recursive calls are needed for this

# Top-down Approach

> **Require:** Integer $n$, Array $p$ of length $n$ with prices
> Let $r[0 \dots n]$ be a new array
> **for** $i = 0 \dots n$ **do**
> $r[i] \leftarrow -\infty$
> **return** $\textsc{Memoized-Cut-Pole-Aux}(p, n, r)$

Algorithm $\textsc{Memoized-Cut-Pole}(p,\ n)$

- Prepare a table $r$ of size $n$
- Initialize all elements of $r$ with $-\infty$
- Actual work is done in $\textsc{Memoized-Cut-Pole-Aux}$, table $r$ is passed on to $\textsc{Memoized-Cut-Pole-Aux}$

# Top-down Approach (2)

**Require:** Integer $n$, array $p$ of length $n$ with prices, array $r$ of revenues

```
if r[n] ≥ 0 then
   return r[n]
if n = 0 then
   q ← 0
else
   q ← −∞
   for i = 1 ... n do
      q ← max{q, p[i] + MEMOIZED-CUT-POLE-AUX(p, n − i, r)}
r[n] ← q
return q
```

Algorithm $\textsc{Memoized-Cut-Pole-Aux}(p,\ n,\ r)$

**Observe:** If $r[n] \geq 0$ then $r[n]$ has been computed previously

# Bottom-up Approach

**Require:** Integer $n$, array $p$ of length $n$ with prices

```
Let r[0...n] be a new array
r[0] ← 0
for j = 1...n do
    q ← −∞
    for i = 1...j do
        q ← max{q, p[i] + r[j − i]}
    r[j] ← q
return r[n]
```

Algorithm $\textsc{Bottom-Up-Cut-Pole}(p, n)$

**Runtime:** Two nested for-loops

$$\sum_{j=1}^{n} \sum_{i=1}^{j} O(1) = O(1) \sum_{j=1}^{n} \sum_{i=1}^{j} 1 = O(1) \sum_{j=1}^{n} j = O(1) \frac{n(n+1)}{2} = O(n^2) \ .$$

# Conclusion

**Runtime of Top-down Approach** $O(n^2)$

(please think about this!)

**Dynamic Programming**

- Solves a problem by combining subproblems
- Subproblems are solved at most once, store solutions in table
- If a problem exhibits *optimal substructure* then dynamic programming is often the right choice
- Top-down and bottom-up approaches have the same runtime