

Video 22: Dynamic Programming - Matrix Chain Parenthesization

COMS10017 - (Object-Oriented Programming and) Algorithms

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Matrix Multiplication

Problem: MATRIX-MULTIPLICATION

- 1 **Input:** Matrices A, B with $A.\text{columns} = B.\text{rows}$
- 2 **Output:** Matrix product $A \times B$

Example:

$$p \begin{matrix} & q \\ \begin{pmatrix} 2 & 3 \\ 1 & 0 \\ 2 & 6 \\ 0 & 9 \end{pmatrix} & \times \begin{matrix} r \\ q \\ \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 0 \end{pmatrix} \end{matrix} = \begin{matrix} r \\ p \\ \begin{pmatrix} 6 & 2 & 4 \\ 0 & 1 & 2 \\ 12 & 2 & 4 \\ 18 & 0 & 0 \end{pmatrix} \end{matrix}$$

Notation: $p \times q$ matrix: p rows and q columns

- $p \times q$ matrix times $q \times r$ matrix gives a $p \times r$ matrix
- $(A \times B)_{i,j} = \text{row } i \text{ of } A \text{ times column } j \text{ of } B$

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- $(A \times B)_{i,j} = \text{row } i \text{ of } A \text{ times column } j \text{ of } B$

Algorithm for Matrix-Multiplication

Algorithm: $(A \times B)_{i,j}$ = row i of A times column j of B

Require: Matrices A, B with $A.columns = B.rows$

Let C be a new $A.rows \times B.columns$ matrix

for $i \leftarrow 1 \dots A.rows$ **do**

for $j \leftarrow 1 \dots B.columns$ **do**

$C_{ij} \leftarrow 0$

for $k \leftarrow 1 \dots A.columns$ **do**

$C_{ij} \leftarrow C_{ij} + A_{ik} \cdot B_{kj}$

return C

Algorithm MATRIX-MULTIPLY(A, B)

Runtime:

- Three nested loops: $O(A.rows \cdot B.columns \cdot A.columns)$
- Number of Multiplications: $A.rows \cdot B.columns \cdot A.columns$
- Multiplying two $n \times n$ matrices: runtime $O(n^3)$

History: Multiplying two $n \times n$ matrices

- before 1969: $O(n^3)$
- 1969: Strassen $O(n^{2.8074})$ (divide-and-conquer)
- 1990: Coppersmith-Winograd $O(n^{2.3755})$
- 2010: Stothers $O(n^{2.374})$
- 2011: Virginia Williams $O(n^{2.3728642})$
- 2014: Le Gall $O(n^{2.3728639})$

Important Problem:

- Many algorithms rely on fast matrix multiplication
- Better bound for matrix multiplication improves many algorithms

The Matrix-chain Multiplication Problem

Problem: MATRIX-CHAIN-MULTIPLICATION

- 1 **Input:** A sequence (chain) of n matrices $A_1, A_2, A_3, \dots, A_n$
- 2 **Output:** The product $A_1 \times A_2 \times A_3 \times \dots \times A_n$

Discussion:

- $A_i.\text{columns} = A_{i+1}.\text{rows}$ for every $1 \leq i < n$
- Assume A_i has dimension $p_{i-1} \times p_i$, for vector $p[0 \dots n]$
- Matrix product is associative:

$$(A_1 \times A_2) \times A_3 = A_1 \times (A_2 \times A_3)$$

Exploit Associativity: Parenthesize $A_1 \times A_2 \times A_3 \times \dots \times A_n$ so as to minimize the number of scalar multiplications (and thus the runtime)

Example: Three matrices A_1, A_2, A_3 with dimensions

$$A_1 : 10 \times 100 \quad A_2 : 100 \times 5 \quad A_3 : 5 \times 50$$

($p_0 = 10, p_1 = 100, p_2 = 5, p_3 = 50$)

Computation of $(A_1 \times A_2) \times A_3$:

- $A_1 \times A_2 = A_{12}$ requires $10 \cdot 100 \cdot 5 = 5000$ multiplications
- $A_{12} \times A_3$ requires $10 \cdot 5 \cdot 50 = 2500$ multiplications
- Total: 7500 multiplications

Computation of $A_1 \times (A_2 \times A_3)$:

- $A_2 \times A_3 = A_{23}$ requires $100 \cdot 5 \cdot 50 = 25000$ multiplications
- $A_1 \times A_{23}$ requires $10 \cdot 100 \cdot 50 = 50000$ multiplications
- Total: 75000 multiplications

The Matrix-Chain-Parenthesization Problem

Problem: MATRIX-CHAIN-PARENTHEZIZATION

- 1 **Input:** A sequence (chain) of n matrices $A_1, A_2, A_3, \dots, A_n$
- 2 **Output:** A parenthesization of $A_1 \times A_2 \times A_3 \times \dots \times A_n$ that minimizes the number of scalar multiplications

How many Parenthesizations $P(n)$ are there?

- We write: A_{ij} for the product $A_i \times A_{i+1} \times \dots \times A_j$
- There is a final matrix multiplication: $A_{1k} \times A_{(k+1)n}$, for some $1 \leq k \leq n - 1$. Hence:

$$P(n) = \begin{cases} 1 & \text{if } n = 1, \\ \sum_{k=1}^{n-1} P(k)P(n-k) & \text{if } n \geq 2. \end{cases}$$

Example: Four matrices A_1, A_2, A_3, A_4

$$A_1 \times A_{24} \quad A_{12} \times A_{34} \quad A_{13} \times A_4$$

Number of Parenthesizations

Example (continued): Four matrices A_1, A_2, A_3, A_4

$$A_1 \times A_{24} \quad A_{12} \times A_{34} \quad A_{13} \times A_4$$

$$P(3) = \sum_{k=1}^2 P(k)P(n-k) = P(1)P(2) + P(2)P(1) = 2$$

$$\begin{aligned} P(4) &= \sum_{k=1}^3 P(k)P(n-k) = P(1)P(3) + P(2)P(2) + P(3)P(1) \\ &= P(3) + 1 + P(3) = 2P(3) + 1 = 5 . \end{aligned}$$

- 1 $A_1 \times ((A_2 \times A_3) \times A_4)$
- 2 $A_1 \times ((A_2 \times (A_3 \times A_4)))$
- 3 $(A_1 \times A_2) \times (A_3 \times A_4)$
- 4 $((A_1 \times A_2) \times A_3) \times A_4$
- 5 $(A_1 \times (A_2 \times A_3)) \times A_4$

A Bound on the Number of Parenthesizations:

$$P(n) = \begin{cases} 1 & \text{if } n = 1, \\ \sum_{k=1}^{n-1} P(k)P(n-k) & \text{if } n \geq 2. \end{cases}$$

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, ...

- It can be seen that there are $\Omega(2^n)$ possibilities
- An efficient algorithm thus cannot try out all possibilities

Dynamic Programming!

Optimal Substructure

We say that a problem P exhibits *optimal substructure* if:

An optimal solution to P contains within it optimal solutions to subproblems of P .

Optimal Substructure in Matrix-Chain-Parentesization

- Consider optimal solution to instance of size n
- Suppose that last product is $A_{1k} \times A_{(k+1)n}$
- Then the optimal solution contains optimal parenthesizations of $A_1 \times A_2 \times \dots \times A_k$ and $A_{k+1} \times A_{k+2} \times \dots \times A_n$

Proof. Suppose it did not contain optimal parenthesizations of $A_1 \times A_2 \times \dots \times A_k$ and of $A_{k+1} \times A_{k+2} \times \dots \times A_n$. Then picking optimal parenthesizations of the two subproblems would give better solution to initial instance. □

Optimal Solution to Subproblem:

- $m[i, j]$: minimum number of scalar multiplications needed to compute $A_i \times A_{i+1} \times \cdots \times A_j = A_{ij}$
- Observe that $m[i, i] = 0$ (chain consists of single matrix A_i)
- Suppose $j > i$. Suppose last multiplication in optimal solution is: $A_{ik} \times A_{(k+1)j}$, for some k

- Then: cost of multiplying $A_{ik} \times A_{(k+1)j}$
$$m[i, j] = m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$$

(A_{ik} : $p_{i-1} \times p_k$ matrix, $A_{(k+1)j}$: $p_k \times p_j$ matrix)

- Since we do not know k , we try out all possibilities and choose the best solution:

$$m[i, j] = \begin{cases} 0 & \text{if } i = j, \\ \min_{i \leq k < j} \{m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j\} & \text{if } i < j. \end{cases}$$

$$m[i, j] = \begin{cases} 0 & \text{if } i = j, \\ \min_{i \leq k < j} \{m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j\} & \text{if } i < j. \end{cases}$$

Algorithmic Considerations:

- As in POLE-CUTTING, we could implement this recursive formula directly. \rightarrow exponential runtime
- Instead, we compute the table $m[i, j]$ bottom up
- Observe that there are less than n^2 subproblems $m[i, j]$ (i and j take values in $\{1, \dots, n\}$)
- We will see that computing one value $m[i, j]$ takes $O(n)$ time
- This yields an $O(n^3)$ time algorithm

Dynamic Programming Algorithm

```
Require: Integer  $n$ , vector of dimensions of matrices  $p$  so that  
matrix  $A_i$  has dimensions  $p_{i-1} \times p_i$   
Let  $m[1 \dots n, 1 \dots n]$  be a new array  
for  $i \leftarrow 1 \dots n$  do  
     $m[i, i] \leftarrow 0$   
for  $\ell \leftarrow 2 \dots n$  do {chain length}  
    for  $i \leftarrow 1 \dots n - \ell + 1$  do {left position}  
         $j \leftarrow i + \ell - 1$  {right position}  
         $m[i, j] \leftarrow \infty$   
        for  $k \leftarrow i \dots j - 1$  do  
             $m[i, j] \leftarrow \min\{m[i, j], m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j\}$   
return  $m$ 
```

Algorithm MATRIX-CHAIN-VALUE(n, p)

Runtime: $O(n^3)$ (by evaluating $\sum_{\ell=2}^n \sum_{i=1}^{n-\ell+1} \sum_{k=1}^{i+\ell-2} O(1)$)

Useful Formula:

$$\sum_{i=a}^b 1 = b - a + 1$$

$$\begin{aligned} & \sum_{l=2}^n \sum_{i=1}^{n-l+1} \sum_{k=1}^{i+l-2} O(1) = O(1) \cdot \sum_{l=2}^n \sum_{i=1}^{n-l+1} \sum_{k=1}^{i+l-2} 1 \\ & \leq O(1) \cdot \sum_{l=1}^n \sum_{i=1}^n \sum_{k=1}^n 1 = O(1) \cdot \sum_{l=1}^n \sum_{i=1}^n n = O(1) \cdot n \sum_{l=1}^n \sum_{i=1}^n 1 \\ & = O(1) \cdot n \sum_{l=1}^n n = O(1) \cdot n^2 \sum_{l=1}^n 1 = O(1) \cdot n^2 \cdot n = O(1)n^3 \\ & = O(n^3). \end{aligned}$$

Example $n = 4$ and $p = 3 \ 7 \ 6 \ 2 \ 9$

	1	2	3	4
1				
2				
3				
4				

```
for  $i \leftarrow 1 \dots n$  do  
   $m[i, i] \leftarrow 0$ 
```


Example $n = 4$ and $p = 3 \ 7 \ 6 \ 2 \ 9$

	1	2	3	4
1	0			
2		0		
3			0	
4				0

```
for  $i \leftarrow 1 \dots n$  do  
   $m[i, i] \leftarrow 0$ 
```

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1	0			
2		0		
3			0	
4				0

```

for  $l \leftarrow 2 \dots n$  do
  for  $i \leftarrow 1 \dots n - l + 1$  do {left position}
     $j \leftarrow i + l - 1$  {right position}
     $m[i, j] \leftarrow \infty$ 
    for  $k \leftarrow i \dots j - 1$  do
       $m[i, j] \leftarrow \min\{m[i, j], m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j\}$ 

```

$l = 2, i = 1, j = 2$

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1	0			
2	126	0		
3			0	
4				0

for $l \leftarrow 2 \dots n$ **do**

for $i \leftarrow 1 \dots n - l + 1$ **do** {left position}

$j \leftarrow i + l - 1$ {right position}

$m[i, j] \leftarrow \infty$

for $k \leftarrow i \dots j - 1$ **do**

$m[i, j] \leftarrow \min\{m[i, j], m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j\}$

$l = 2, i = 1, j = 2$

$$m[1, 2] = m[1, 1] + m[2, 2] + p_0p_1p_2 = 0 + 0 + 3 \cdot 7 \cdot 6 = 126$$

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1	0			
2	126	0		
3			0	
4				0

```

for  $l \leftarrow 2 \dots n$  do
  for  $i \leftarrow 1 \dots n - l + 1$  do {left position}
     $j \leftarrow i + l - 1$  {right position}
     $m[i, j] \leftarrow \infty$ 
    for  $k \leftarrow i \dots j - 1$  do
       $m[i, j] \leftarrow \min\{m[i, j], m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j\}$ 

```

$l = 2, i = 2, j = 3$

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1	0			
2	126	0		
3		84	0	
4				0

```

for  $l \leftarrow 2 \dots n$  do
  for  $i \leftarrow 1 \dots n - l + 1$  do {left position}
     $j \leftarrow i + l - 1$  {right position}
     $m[i, j] \leftarrow \infty$ 
    for  $k \leftarrow i \dots j - 1$  do
       $m[i, j] \leftarrow \min\{m[i, j], m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j\}$ 

```

$$l = 2, i = 2, j = 3$$

$$m[2, 3] = m[2, 2] + m[3, 3] + p_1 p_2 p_3 = 0 + 0 + 7 \cdot 6 \cdot 2 = 84$$

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1	0			
2	126	0		
3		84	0	
4				0

for $l \leftarrow 2 \dots n$ **do**

for $i \leftarrow 1 \dots n - l + 1$ **do** {left position}

$j \leftarrow i + l - 1$ {right position}

$m[i, j] \leftarrow \infty$

for $k \leftarrow i \dots j - 1$ **do**

$m[i, j] \leftarrow \min\{m[i, j], m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j\}$

$l = 2, i = 3, j = 4$

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1	0			
2	126	0		
3		84	0	
4			108	0

for $l \leftarrow 2 \dots n$ **do**

for $i \leftarrow 1 \dots n - l + 1$ **do** {left position}

$j \leftarrow i + l - 1$ {right position}

$m[i, j] \leftarrow \infty$

for $k \leftarrow i \dots j - 1$ **do**

$m[i, j] \leftarrow \min\{m[i, j], m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j\}$

$l = 2, i = 3, j = 4$

$$m[3, 4] = m[3, 3] + m[4, 4] + p_2 p_3 p_4 = 0 + 0 + 6 \cdot 2 \cdot 9 = 108$$

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1	0			
2	126	0		
3		84	0	
4			108	0

for $l \leftarrow 2 \dots n$ **do**

for $i \leftarrow 1 \dots n - l + 1$ **do** {left position}

$j \leftarrow i + l - 1$ {right position}

$m[i, j] \leftarrow \infty$

for $k \leftarrow i \dots j - 1$ **do**

$m[i, j] \leftarrow \min\{m[i, j], m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j\}$

$l = 3, i = 1, j = 3$

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1	0			
2	126	0		
3	106	84	0	
4			108	0

```

for  $l \leftarrow 2 \dots n$  do
  for  $i \leftarrow 1 \dots n - l + 1$  do {left position}
     $j \leftarrow i + l - 1$  {right position}
     $m[i, j] \leftarrow \infty$ 
    for  $k \leftarrow i \dots j - 1$  do
       $m[i, j] \leftarrow \min\{m[i, j], m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j\}$ 

```

$$l = 3, i = 1, j = 3$$

$$m[1, 1] + m[2, 3] + p_0 p_1 p_3 = 0 + 84 + 3 \cdot 7 \cdot 2 = 84 + 42 = 106$$

$$m[1, 2] + m[3, 3] + p_0 p_2 p_3 = 126 + 0 + 3 \cdot 6 \cdot 2 = 126 + 36 = 162$$

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1	0			
2	126	0		
3	106	84	0	
4			108	0

for $l \leftarrow 2 \dots n$ **do**

for $i \leftarrow 1 \dots n - l + 1$ **do** {left position}

$j \leftarrow i + l - 1$ {right position}

$m[i, j] \leftarrow \infty$

for $k \leftarrow i \dots j - 1$ **do**

$m[i, j] \leftarrow \min\{m[i, j], m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j\}$

$l = 3, i = 2, j = 4$

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1	0			
2	126	0		
3	106	84	0	
4		210	108	0

for $l \leftarrow 2 \dots n$ **do**

for $i \leftarrow 1 \dots n - l + 1$ **do** {left position}

$j \leftarrow i + l - 1$ {right position}

$m[i, j] \leftarrow \infty$

for $k \leftarrow i \dots j - 1$ **do**

$m[i, j] \leftarrow \min\{m[i, j], m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j\}$

$l = 3, i = 2, j = 4$

$$m[2, 2] + m[3, 4] + p_1 p_2 p_4 = 0 + 108 + 7 \cdot 6 \cdot 9 = 108 + 378 = 486$$

$$m[2, 3] + m[4, 4] + p_1 p_3 p_4 = 84 + 0 + 7 \cdot 2 \cdot 9 = 84 + 36 = \mathbf{210}$$

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1	0			
2	126	0		
3	106	84	0	
4		210	108	0

for $l \leftarrow 2 \dots n$ **do**

for $i \leftarrow 1 \dots n - l + 1$ **do** {left position}

$j \leftarrow i + l - 1$ {right position}

$m[i, j] \leftarrow \infty$

for $k \leftarrow i \dots j - 1$ **do**

$m[i, j] \leftarrow \min\{m[i, j], m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j\}$

$l = 4, i = 1, j = 4$

Example $n = 4$ and $p = 3 \quad 7 \quad 6 \quad 2 \quad 9$

	1	2	3	4
1	0			
2	126	0		
3	106	84	0	
4	160	210	108	0

for $l \leftarrow 2 \dots n$ **do**

for $i \leftarrow 1 \dots n - l + 1$ **do** {left position}

$j \leftarrow i + l - 1$ {right position}

$m[i, j] \leftarrow \infty$

for $k \leftarrow i \dots j - 1$ **do**

$m[i, j] \leftarrow \min\{m[i, j], m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j\}$

$$m[1, 1] + m[2, 4] + p_0p_1p_4 = 0 + 210 + 3 \cdot 7 \cdot 9 = 399$$

$$m[1, 2] + m[3, 4] + p_0p_2p_4 = 126 + 108 + 3 \cdot 6 \cdot 9 = 396$$

$$m[1, 3] + m[4, 4] + p_0p_3p_4 = 106 + 0 + 3 \cdot 2 \cdot 9 = \mathbf{160}$$

Optimal Solution of Example

Example: $n = 4$ and $p = 3 \ 7 \ 6 \ 2 \ 9$

- Algorithm outputs value of optimal solution: $m[1, 4] = 160$
- We would like to know the optimal parenthesization as well

$$((A_1 \times A_2) \times A_3) \times A_4$$

→ Modify algorithm to keep track of parameters that give minimum in array s

Keep Track of Optimal Choices

```
Require: Integer  $n$ , vector of dimensions of matrices  $p$  so that  
matrix  $A_i$  has dimensions  $p_{i-1} \times p_i$   
Let  $m[1 \dots n, 1 \dots n]$  be a new array  
for  $i \leftarrow 1 \dots n$  do  
     $m[i, i] \leftarrow 0$   
for  $l \leftarrow 2 \dots n$  do {chain length}  
    for  $i \leftarrow 1 \dots n - l + 1$  do {left position}  
         $j \leftarrow i + l - 1$  {right position}  
         $m[i, j] \leftarrow \infty$   
        for  $k \leftarrow i \dots j - 1$  do  
             $m[i, j] \leftarrow \min\{m[i, j], m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j\}$   
return  $m, s$ 
```

Algorithm MATRIX-CHAIN-VALUE(n, p)

Keep Track of Optimal Choices

```
Require: Integer  $n$ , vector of dimensions of matrices  $p$  so that  
matrix  $A_i$  has dimensions  $p_{i-1} \times p_i$   
Let  $m[1 \dots n, 1 \dots n]$  and  $s[1 \dots n, 2 \dots n]$  be new arrays  
for  $i \leftarrow 1 \dots n$  do  
   $m[i, i] \leftarrow 0$   
for  $l \leftarrow 2 \dots n$  do {chain length}  
  for  $i \leftarrow 1 \dots n - l + 1$  do {left position}  
     $j \leftarrow i + l - 1$  {right position}  
     $m[i, j] \leftarrow \infty$   
    for  $k \leftarrow i \dots j - 1$  do  
       $q \leftarrow m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$   
      if  $q < m[i, j]$  then  
         $m[i, j] \leftarrow q$   
         $s[i, j] \leftarrow k$   
return  $m$ 
```

Algorithm MATRIX-CHAIN-ORDER(A, B)

Using s to find Optimal Parenthesization

```
Require: Array  $s$ , positions  $i, j$   
  if  $i = j$  then  
    print " $A_i$ "  
  else  
    print "("  
    PRINT-OPTIMAL-PARENS( $s, i, s[i, j]$ )  
    PRINT-OPTIMAL-PARENS( $s, s[i, j] + 1, j$ )  
    print ")"
```

Algorithm PRINT-OPTIMAL-PARENS(s, i, j)

Call PRINT-OPTIMAL-PARENS($s, 1, n$) to obtain parenthesization