Video 24: Elements of Dynamic Programming

COMS10017 - (Object-Oriented Programming and) Algorithms

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Elements of Dynamic Programming

Solving a Problem with Dynamic Programming:

1. Identify optimal substructure
2. Give recursive solution
3. Compute optimal costs
4. Construct optimal solution

Discussion:

- Steps 1 and 2 requires studying the problem at hand
- Steps 3 and 4 are usually straightforward

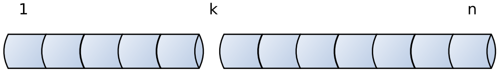
Step 1: Identify Optimal Substructure

Optimal Substructure Problem P exhibits *optimal substructure* if:

An optimal solution to P contains within it optimal solutions to subproblems of P.

Examples: Let OPT be optimal solution

- POLE-CUTTING: If OPT cuts at position k then cuts within $\{1, \dots, k-1\}$ form opt. solution to pole of len. k, and cuts within $\{k+1, \dots, n\}$ form opt. solution to pole of len. $n-k$.



- MATRIX-CHAIN-PARENTHESIZATION: If in OPT final multiplication is $A_{1k} \times A_{(k+1)n}$ then OPT contains optimal parenthesizations of $A_1 \times \cdots \times A_k$ and $A_{k+1} \times \cdots \times A_n$

$$(A_1 \times (A_2 \times A_3)) \times ((A_4 \times A_5) \times A_6)$$

Step 2. Give Recursive Solution

Define Table for Storing Optimal Solutions to Subproblems:
Optimal substructure indicates how subproblems look like

- Pole-Cutting:
  OPT contains optimal solutions to shorter lengths
  $\rightarrow$ Store optimal solutions for every length in $\{1, \ldots, n\}$
  (table of length n)

- Matrix-Chain-Parenthesization:
  OPT contains optimal parenthesizations for subproducts
  $A_i \times \cdots \times A_j$
  $\rightarrow$ Store optimal parenthesizations for every subproduct
  $A_i \times \cdots \times A_j$ (table of size n^2)

Step 2. Give Recursive Solution (2)

Express Optimal Solutions Recursively:

- POLE-CUTTING: (p_k: price for selling a pole of length k)

$$m[i] := \text{maximum revenue to pole of length } i$$

$$m[i] = \max_{1 \le k \le i} p_k + m_{i-k}$$

- MATRIX-CHAIN-PARENTHESIZATION:

$$m[i,j] := \text{min. \# scalar mult. to compute } A_i \times A_{i+1} \times \cdots \times A_j$$

$$\begin{aligned} m[i,j] &= \min_{i \le k < j} m[i,k] + m[k+1,j] \\ &+ \text{“cost for computing } A_{ik} \times A_{(k+1)j}\text{”} \end{aligned}$$

Compute Optimal Costs

Two Possibilities:

- Bottom-up
- Top-down with memoization

Example: Bottom-up for POLE-CUTTING

length i	1	2	3	4	5	6	7	8	9	10
price $p(i)$	1	5	8	9	10	17	17	20	24	30

$$m[i] = \max_{1 \leq k \leq i} p_k + m_{i-k}$$

0	1	2	3	4	5	6	7	8	9	10
-	-	-	-	-	-	-	-	-	-	-

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length i	1	2	3	4	5	6	7	8	9	10
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$$m[i] = \max_{1 \leq k \leq i} p_k + m_{i-k}$$

0	1	2	3	4	5	6	7	8	9	10
-	-	-	-	-	-	-	-	-	-	-

Initialize base cases: $m[0] = 0$ and $m[1] = p_1$

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- Top-down with memoization

Example: Bottom-up for POLE-CUTTING

length i	1	2	3	4	5	6	7	8	9	10
price $p(i)$	1	5	8	9	10	17	17	20	24	30

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0	1	2	3	4	5	6	7	8	9	10
0	1	-	-	-	-	-	-	-	-	-

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price $p(i)$	1	5	8	9	10	17	17	20	24	30

$$m[i] = \max_{1 \le k \le i} p_k + m_{i-k}$$

0	1	2	3	4	5	6	7	8	9	10
0	1	-	-	-	-	-	-	-	-	-

$$m[2] = \max\{p_1 + m_1, p_2 + m_0\} = \max\{1+1, 5+0\} = 5$$

Compute Optimal Costs

Two Possibilities:

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Example: Bottom-up for POLE-CUTTING

length i	1	2	3	4	5	6	7	8	9	10
price $p(i)$	1	5	8	9	10	17	17	20	24	30

$$m[i] = \max_{1 \le k \le i} p_k + m_{i-k}$$

0	1	2	3	4	5	6	7	8	9	10
0	1	5	-	-	-	-	-	-	-	-

$m[2] = \max\{p_1 + m_1, p_2 + m_0\} = \max\{1 + 1, 5 + 0\} = 5$

Compute Optimal Costs

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Example: Bottom-up for POLE-CUTTING

length i	1	2	3	4	5	6	7	8	9	10
price $p(i)$	1	5	8	9	10	17	17	20	24	30

$$m[i] = \max_{1 \leq k \leq i} p_k + m_{i-k}$$

0	1	2	3	4	5	6	7	8	9	10
0	1	5	-	-	-	-	-	-	-	-

$$m[3] = \max\{p_1+m_2, p_2+m_1, p_3+m_0\} = \max\{1+5, 5+1, 8+0\} = 8$$

Compute Optimal Costs

Two Possibilities:

- Bottom-up
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Example: Bottom-up for POLE-CUTTING

length i	1	2	3	4	5	6	7	8	9	10
price $p(i)$	1	5	8	9	10	17	17	20	24	30

$$m[i] = \max_{1 \le k \le i} p_k + m_{i-k}$$

0	1	2	3	4	5	6	7	8	9	10
0	1	5	8	-	-	-	-	-	-	-

$m[3] = \max\{p_1+m_2, p_2+m_1, p_3+m_0\} = \max\{1+5, 5+1, 8+0\} = 8$

Compute Optimal Costs

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Example: Bottom-up for POLE-CUTTING

length i	1	2	3	4	5	6	7	8	9	10
price $p(i)$	1	5	8	9	10	17	17	20	24	30

$$m[i] = \max_{1 \le k \le i} p_k + m_{i-k}$$

0	1	2	3	4	5	6	7	8	9	10
0	1	5	8	-	-	-	-	-	-	-

$m[4] = \max\{p_1 + m_3, p_2 + m_2, p_3 + m_1, p_4 + m_0\} = \max\{1 + 8, 5 + 5, 8 + 1, 9\} = 10$

Compute Optimal Costs

Two Possibilities:

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Example: Bottom-up for POLE-CUTTING

length i	1	2	3	4	5	6	7	8	9	10
price $p(i)$	1	5	8	9	10	17	17	20	24	30

$$m[i] = \max_{1 \leq k \leq i} p_k + m_{i-k}$$

0	1	2	3	4	5	6	7	8	9	10
0	1	5	8	10	-	-	-	-	-	-

$m[4] = \max\{p_1 + m_3, p_2 + m_2, p_3 + m_1, p_4 + m_0\} = \max\{1 + 8, 5 + 5, 8 + 1, 9\} = 10$

Compute Optimal Costs

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length i	1	2	3	4	5	6	7	8	9	10
price $p(i)$	1	5	8	9	10	17	17	20	24	30

$$m[i] = \max_{1 \leq k \leq i} p_k + m_{i-k}$$

0	1	2	3	4	5	6	7	8	9	10
0	1	5	8	10	-	-	-	-	-	-

$m[5] = \max\{p_1 + m_4, p_2 + m_3, p_3 + m_2, p_4 + m_1, p_5 + m_0\} = \max\{1 + 10, 5 + 8, 8 + 2, 9 + 1, 10\} = 13$

Compute Optimal Costs

Two Possibilities:

- Bottom-up
- Top-down with memoization

Example: Bottom-up for Pole-Cutting

length i	1	2	3	4	5	6	7	8	9	10
price $p(i)$	1	5	8	9	10	17	17	20	24	30

$$m[i] = \max_{1 \leq k \leq i} p_k + m_{i-k}$$

0	1	2	3	4	5	6	7	8	9	10
0	1	5	8	10	13	-	-	-	-	-

$m[5] = \max\{p_1 + m_4, p_2 + m_3, p_3 + m_2, p_4 + m_1, p_5 + m_0\} = \max\{1 + 10, 5 + 8, 8 + 2, 9 + 1, 10\} = 13$

Compute Optimal Costs

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Example: Bottom-up for POLE-CUTTING

length i	1	2	3	4	5	6	7	8	9	10
price $p(i)$	1	5	8	9	10	17	17	20	24	30

$$m[i] = \max_{1 \le k \le i} p_k + m_{i-k}$$

0	1	2	3	4	5	6	7	8	9	10
0	1	5	8	10	13	-	-	-	-	-

...

Compute Optimal Costs

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0	1	5	8	10	13	17	18	22	25	30

Compute Optimal Costs

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Example: Bottom-up for POLE-CUTTING

length i	1	2	3	4	5	6	7	8	9	10
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$$m[i] = \max_{1 \le k \le i} p_k + m_{i-k}$$

0	1	2	3	4	5	6	7	8	9	10
0	1	5	8	10	13	17	18	22	25	30

The maximum revenue obtainable for a pole of length 10 is 30

Compute Optimal Costs

Two Possibilities:

- Bottom-up
- Top-down with memoization

Example: Bottom-up for POLE-CUTTING

length i	1	2	3	4	5	6	7	8	9	10
price $p(i)$	1	5	8	9	10	17	17	20	24	30

$$m[i] = \max_{1 \le k \le i} p_k + m_{i-k}$$

0	1	2	3	4	5	6	7	8	9	10
0	1	5	8	10	13	17	18	22	25	30

But how can we find out how to cut the pole?

Step 4: Construct Optimal Solution

Keep Track of Optimal Choices: store optimal choices in array s

Require: Integer n, array p of length n with prices
Let $r[0 \dots n]$ be a new array
$r[0] \leftarrow 0$
for $j = 1 \dots n$ **do**
$r[j] \leftarrow -\infty$
for $i = 1 \dots j$ **do**
$r[j] \leftarrow \max\{r[j], p[i] + r[j-i]\}$
return $r[n]$

Algorithm $\textsc{Bottom-Up-Cut-Pole}(p, n)$

- $s[i]$ contains position of first cut in optimal solution
- Easy to reconstruct all cuts

Step 4: Construct Optimal Solution

Keep Track of Optimal Choices: store optimal choices in array s

```
Require: Integer n, array p of length n with prices
   Let r[0...n] be a new array, let s[1...n] be a new array
   r[0] ← 0
   for j = 1...n do
      r[j] ← −∞
      for i = 1...j do
         if p[i] + r[j − i] > q then
            r[j] ← p[i] + r[j − i]
            s[j] ← i
   return r[n]
```

Algorithm BOTTOM-UP-CUT-POLE(p, n)

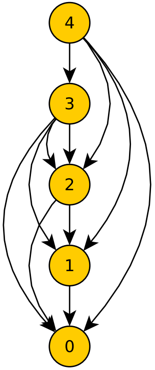
- $s[i]$ contains position of first cut in optimal solution
- Easy to reconstruct all cuts

Subproblem Graph and Runtime

Subproblem Graph

- One node for each subproblem
- Directed edge from a subproblem A to subproblem B if the solution of A depends on the solution of B

Example: Pole-Cutting



Runtime of Dynamic Programming Algorithm:

- Total number of subproblems t
- Maximum number of subproblems a subproblem depends on s
- Runtime: $O(s \cdot t)$ (assuming that computing solution takes time $O(s)$)

Fibonacci Numbers:

$$\begin{aligned} F_0 &= 0, F_1 = 1 \\ F_n &= F_{n-1} + F_{n-2} \text{ for } n \geq 2 \ . \end{aligned}$$

Require: Integer $n \geq 0$
  if $n \leq 1$ **then**
    return n
  else
    $A \leftarrow$ array of size n
    $A[0] \leftarrow 1$, $A[1] \leftarrow 1$
    for $i \leftarrow 2 \dots n$ **do**
      $A[i] \leftarrow A[i-2] + A[i-1]$
    return $A[n]$

$\textsc{DynPrgFib}(n)$

Why is this a dynamic programming algorithm?

Fibonacci Numbers - Dynamic Programming

Identify Optimal Substructure:

- Recall: $F_n = F_{n-1} + F_{n-2}$
- (Optimal) solution to size n problem equals sum of (optimal) solutions to subproblems of sizes $n-1$ and $n-2$ ✓

Give Recursive Solution:

- Recursive solution is already given in the problem description
- $F_n = F_{n-1} + F_{n-2}$

Compute Optimal Costs & Compute Optimal Solution

- Cost and solution is identical for Fibonacci numbers
- There is no need to keep track of optimal choices, since there is only a single choice

Maximum Subarray Problem

Problem: MAXIMUM-SUBARRAY

- **Input:** Array A of n numbers
- **Output:** Indices $0 \leq i \leq j \leq n-1$ such that $\sum_{l=i}^{j} A[l]$ is maximum.

Example:

$$-25 \quad 20 \quad -3 \quad -16 \quad -23 \quad 18 \quad 20 \quad -7 \quad 12 \quad -5 \quad 1$$

Divide-and-Conquer Algorithm

- In lecture 7 we gave a divide-and-conquer algorithm with runtime $O(n \log n)$
- We will give now a faster dynamic programming algorithm

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Example:

$-25 \quad 20 \quad -3 \quad -16 \quad -23 \quad \color{red}{18} \quad \color{red}{20} \quad \color{red}{-7} \quad \color{red}{12} \quad -5 \quad 1$

Divide-and-Conquer Algorithm

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Dynamic Programming for Maximum-Subarray

Related Problem: Maximum-Suffix-Array

- **Input:** Array A of n numbers
- **Output:** Index $0 \leq i \leq n-1$ such that $\sum_{l=i}^{n-1} A[l]$ is maximum.

$$-25 \quad 20 \quad -3 \quad -16 \quad -23 \quad 18 \quad 20 \quad -7 \quad 12 \quad -5 \quad 1$$

Optimal Substructure for Maximum-Subarray:

- Let i, j be the indices of the optimal solution
- Then i is the optimal solution for Maximum-Suffix-Array on input $A[0 \ldots j]$

Dynamic Programming for Maximum-Subarray

Related Problem: Maximum-Suffix-Array

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- **Output:** Index $0 \leq i \leq n-1$ such that $\sum_{l=i}^{n-1} A[l]$ is maximum.

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Dynamic Programming for MAXIMUM-SUBARRAY

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$-25 \quad 20 \quad -3 \quad -16 \quad -23 \quad 18 \quad 20 \quad -7 \quad 12 \quad -5 \quad 1$

Optimal Substructure for MAXIMUM-SUBARRAY:

- Let i, j be the indices of the optimal solution
- Then i is the optimal solution for MAXIMUM-SUFFIX-ARRAY on input $A[0 \dots j]$

$-25 \quad 20 \quad -3 \quad -16 \quad -23 \quad 18 \quad 20 \quad -7 \quad 12 \quad -5 \quad 1$

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$-25 \quad 20 \quad -3 \quad -16 \quad -23 \quad 18 \quad 20 \quad -7 \quad 12 \quad -5 \quad 1$

Optimal Substructure for Maximum-Subarray:

- Let i, j be the indices of the optimal solution
- Then i is the optimal solution for Maximum-Suffix-Array on input $A[0 \dots j]$

$-25 \quad 20 \quad -3 \quad -16 \quad -23 \quad 18 \quad 20 \quad -7 \quad 12$

Dynamic Programming for Maximum Suffix Array

Optimal Substructure:

Lemma

Let A be an array of length n. Let i be the optimal solution for $\textsc{Maximum-Suffix-Array}$ *on A. If $i < n-1$ then the optimal solution to* $\textsc{Maximum-Suffix-Array}$ *on $A[0 \dots n-2]$ is also i.*

$$A[0] \quad A[1] \quad \dots A[i] \quad A[i+1] \quad \dots \quad A[n-2] \quad A[n-1]$$

Proof. Suppose that the lemma is not true and suppose that $i' \neq i$ is the optimal solution to $\textsc{Maximum-Suffix-Array}$ on $A[0 \dots n-2]$. Then,

$$\sum_{j=i'}^{n-2} A[j] > \sum_{j=i}^{n-2} A[j]$$

But then $\sum_{j=i'}^{n-1} A[j] > \sum_{j=i}^{n-1} A[j]$, a contradiction to the fact that i is optimal for A. □

Recursive Solution to Maximum Suffix Array

Recursive Solution:

$$m[i] := \text{value of maximum suffix array of } A[0 \ldots i]$$

$$m[i] = \begin{cases} A[0] & \text{if } i = 0 \\ A[i] & \text{if } m[i-1] \leq 0 \\ m[i-1] + A[i] & \text{if } m[i-1] > 0 \ . \end{cases}$$

Example: Bottom-up Computation

	0	1	2	3	4	5	6	7	8	9	10
A	-25	20	-3	-16	-23	18	20	-7	12	-5	1
m											

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Example: Bottom-up Computation

	0	1	2	3	4	5	6	7	8	9	10
A	-25	20	-3	-16	-23	18	20	-7	12	-5	1
m	-25										

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	0	1	2	3	4	5	6	7	8	9	10
A	-25	20	-3	-16	-23	18	20	-7	12	-5	1
m	-25	20									

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Example: Bottom-up Computation

	0	1	2	3	4	5	6	7	8	9	10
A	-25	20	-3	-16	-23	18	20	-7	12	-5	1
m	-25	20	17								

Recursive Solution to Maximum Suffix Array

Recursive Solution:

$$m[i] := \text{value of maximum suffix array of } A[0 \ldots i]$$

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Example: Bottom-up Computation

	0	1	2	3	4	5	6	7	8	9	10
A	-25	20	-3	-16	-23	18	20	-7	12	-5	1
m	-25	20	17	1							

Recursive Solution:

$$m[i] := \text{value of maximum suffix array of } A[0 \dots i]$$

$$m[i] = \begin{cases} A[0] & \text{if } i = 0 \\ A[i] & \text{if } m[i-1] \leq 0 \\ m[i-1] + A[i] & \text{if } m[i-1] > 0 \ . \end{cases}$$

Example: Bottom-up Computation

	0	1	2	3	4	5	6	7	8	9	10
A	-25	20	-3	-16	-23	18	20	-7	12	-5	1
m	-25	20	17	1	-22						

Recursive Solution to Maximum Suffix Array

Recursive Solution:

$$m[i] := \text{value of maximum suffix array of } A[0 \ldots i]$$

$$m[i] = \begin{cases} A[0] & \text{if } i = 0 \\ A[i] & \text{if } m[i-1] \leq 0 \\ m[i-1] + A[i] & \text{if } m[i-1] > 0 \,. \end{cases}$$

Example: Bottom-up Computation

	0	1	2	3	4	5	6	7	8	9	10
A	-25	20	-3	-16	-23	18	20	-7	12	-5	1
m	-25	20	17	1	-22	18					

Recursive Solution to Maximum Suffix Array

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$$m[i] := \text{value of maximum suffix array of } A[0 \ldots i]$$

$$m[i] = \begin{cases} A[0] & \text{if } i = 0 \\ A[i] & \text{if } m[i-1] \leq 0 \\ m[i-1] + A[i] & \text{if } m[i-1] > 0 \ . \end{cases}$$

Example: Bottom-up Computation

	0	1	2	3	4	5	6	7	8	9	10
A	-25	20	-3	-16	-23	18	20	-7	12	-5	1
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Recursive Solution to Maximum Suffix Array

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A	-25	20	-3	-16	-23	18	20	-7	12	-5	1
m	-25	20	17	1	-22	18	38	31			

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$$m[i] = \begin{cases} A[0] & \text{if } i = 0 \\ A[i] & \text{if } m[i-1] \leq 0 \\ m[i-1] + A[i] & \text{if } m[i-1] > 0 \,. \end{cases}$$

Example: Bottom-up Computation

	0	1	2	3	4	5	6	7	8	9	10
A	-25	20	-3	-16	-23	18	20	-7	12	-5	1
m	-25	20	17	1	-22	18	38	31	43		

Recursive Solution to Maximum Suffix Array

Recursive Solution:

$$m[i] := \text{value of maximum suffix array of } A[0 \dots i]$$

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m	-25	20	17	1	-22	18	38	31	43	38	39

Maximum constitutes optimal solution to Maximum-Subarray!

Dynamic Programming Algorithm for Maximum Subarray

Algorithm: Input is an array A of integers of length n

1. Compute dyn. prog. table for MAXIMUM-SUFFIX-ARRAY
2. Return the maximum value in the table

```
Require: Array A of n integers
  Let m[0 ... n − 1] be a new array
  m[0] ← A[0]
  q ← A[0]
  for i = 1 ... n − 1 do
    if m[i − 1] < 0 then
      m[i] ← A[i]
    else
      m[i] ← A[i] + m[i − 1]
    q ← max{q, m[i]}
  return q
```

Kadane's Algorithm for MAXIMUM-SUBARRAY

Summary

Kadane's Algorithm

- Runtime: $O(n)$ (n subproblems, only one subproblem needed to compute current value)
- Recall that Divide-and-Conquer solution has a runtime of $O(n \log n)$
- Observe that for Maximum-Subarray Dynamic Programming and Divide-and-Conquer is applicable

Challenges:

- Compute max. subarray of size at most k, for some k
- Compute subarray $A[i,j]$ such that

$$\frac{\sum_{k=i}^{j} A[k]}{\sqrt{j-i+1}}$$

is maximized.