

# Exercise Sheet 1

## COMS10017 Algorithms 2020/2021

Reminder:  $\log n$  denotes the binary logarithm, i.e.,  $\log n = \log_2 n$ .

### 1 O-notation: Part I

Give formal proofs of the following statements using the definition of Big-O from the lecture (i.e., identify positive constants  $c, n_0$  for which the definition holds):

1.  $5\sqrt{n} \in O(n)$  .
2.  $n^2 + 10n + 8 \in O(\frac{1}{2}n^2)$  .
3.  $n^3 + n^2 + n = O(n^3)$  .
4.  $10 \in O(1)$  .
5.  $\sum_{i=1}^n i \in O(4n^2)$  .

### 2 Racetrack Principle

Use the racetrack principle to prove the following statement:

$$n \leq e^n \text{ holds for every } n \geq 1 .$$

### 3 O-notation: Part II

Give formal proofs of the following statements using the definition of Big-O from the lecture.

1.  $f \in O(h_1), g \in O(h_2)$  then  $f \cdot g \in O(h_1 \cdot h_2)$  .
2.  $2^n \in O(n!)$  .
3.  $2^{\sqrt{\log n}} \in O(n)$  .

### 4 Fast Peak Finding

Consider the following variant of FAST-PEAK-FINDING where the “ $\geq$ ” sign in the condition in instruction 4 is replaced by a “ $<$ ” sign:

1. **if**  $A$  is of length 1 **then return** 0
2. **if**  $A$  is of length 2 **then** compare  $A[0]$  and  $A[1]$  and **return** position of larger element
3. **if**  $A[\lfloor n/2 \rfloor]$  is a peak **then return**  $\lfloor n/2 \rfloor$
4. Otherwise, **if**  $A[\lfloor n/2 \rfloor - 1] < A[\lfloor n/2 \rfloor]$  **then return** FAST-PEAK-FINDING( $A[0, \lfloor n/2 \rfloor - 1]$ )
5. **else return**  $\lfloor n/2 \rfloor + 1 +$  FAST-PEAK-FINDING( $A[\lfloor n/2 \rfloor + 1, n - 1]$ )

Give an input array of length 8 on which this algorithm fails.

## 5 Optional and Difficult

Exercises in this section are intentionally more difficult and are there to challenge yourself.

### 5.1 Advanced Racetrack Principle

Use the racetrack principle and determine a value  $n_0$  such that

$$\frac{2}{\log n} \leq \frac{1}{\log \log n} \text{ holds for every } n \geq n_0 .$$

*Hint:* Transform the inequality and eliminate the log-function from one side of the inequality before applying the racetrack principle. If needed, apply the racetrack principle twice! Recall that  $(\log n)' = \frac{1}{n \ln(2)}$ . The inequality  $\ln(2) \geq 1/2$  may also be useful.

### 5.2 Finding Two Peaks

We are given an integer array  $A$  of length  $n$  that has exactly two peaks. The goal is to find both peaks. We could do this as follows: Simply go through the array with a loop and check every array element. This strategy has a runtime of  $O(n)$  (requires  $c \cdot n$  array accesses, for some constant  $c$ ). Is there a faster algorithm for this problem (e.g. similar to FAST-PEAK-FINDING)? If yes, give such an algorithm. If no, justify why there is no such algorithm.