

Exercise Sheet 4

COMS10017 Algorithms 2020/2021

1 Algorithm Design

Describe an $O(n \log n)$ time algorithm that, given an array A of n integers and another integer x , determines whether or not there are two elements in A whose sum equals x (Hint: Sorting!).

2 Bubblesort

Bubblesort is a popular, but inefficient, sorting algorithm. It works by repeatedly swapping adjacent elements that are out of order:

Algorithm 1 BUBBLESORT

Require: Array A of n integers

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1: for  $i = 0$  to  $n - 2$  do
2:   for  $j = n - 1$  downto  $i + 1$  do
3:     if  $A[j] < A[j - 1]$  then
4:       exchange  $A[j]$  with  $A[j - 1]$ 
5:     end if
6:   end for
7: end for
```

1. What is the worst-case runtime of BUBBLESORT?
2. Consider the loop in lines 2 – 6. Prove that the following invariant holds at the beginning of the loop:

$$A[j] \leq A[k], \text{ for every } k \geq j .$$

Give a suitable termination property of the loop.

3. Consider now the loop in lines 1 – 7. Prove that the following invariant holds at the beginning of the loop:

The subarray $A[0, i]$ is sorted and $A[0, i - 1]$ consists of the $i - 1$ smallest elements of A .

Give a suitable termination property that shows that A is sorted upon termination.

3 Optional and Difficult Questions

Exercises in this section are intentionally more difficult and are there to challenge yourself.

3.1 Proof by Induction

Let n be a positive number that is divisible by 23, i.e., $n = k \cdot 23$, for some integer $k \geq 1$. Let $x = \lfloor n/10 \rfloor$ and let $y = n \% 10$ (the rest of an integer division). Prove by induction on k that 23 divides $x + 7y$.

Example: Consider $k = 4$. Then $n = 92$, $x = 9$ and $y = 2$. Observe that the quantity $x + 7y = 9 + 7 \cdot 2 = 23$ is divisible by 23.