Exercise Sheet 7
COMS10017 Algorithms 2020/2021

Reminder: $\log n$ denotes the binary logarithm, i.e., $\log n = \log_2 n$.

1 Countingsort and Radixsort

1. Illustrate how Countingsort sorts the following array:

4	2	2	0	1	4	2

Solution. See lectures. ✓

2. Illustrate how Radixsort sorts the following binary numbers:

100110 101010 001010 010111 100000 000101

Solution.

$$\begin{array}{ccccccccccccc}
100110 & & 10011\mathbf{0} & & 1000\mathbf{0}0 & & 100\mathbf{0}00 & & 10\mathbf{0}000 & & 1\mathbf{0}0000 & & \mathbf{0}00101 \\
101010 & & 10101\mathbf{0} & & 0001\mathbf{0}1 & & 101\mathbf{0}10 & & 00\mathbf{0}101 & & 0\mathbf{0}0101 & & \mathbf{0}01010 \\
001010 & \rightarrow & 00101\mathbf{0} & \rightarrow & 1001\mathbf{1}0 & \rightarrow & 001\mathbf{0}10 & \rightarrow & 10\mathbf{0}110 & \rightarrow & 1\mathbf{0}0110 & \rightarrow & \mathbf{0}10111 \\
010111 & & 10000\mathbf{0} & & 1010\mathbf{1}0 & & 000\mathbf{1}01 & & 01\mathbf{0}111 & & 1\mathbf{0}1010 & & \mathbf{1}00000 \\
100000 & & 01011\mathbf{1} & & 0010\mathbf{1}0 & & 100\mathbf{1}10 & & 10\mathbf{1}010 & & 0\mathbf{0}1010 & & \mathbf{1}00110 \\
000101 & & 00010\mathbf{1} & & 0101\mathbf{1}1 & & 010\mathbf{1}11 & & 00\mathbf{1}010 & & 0\mathbf{1}0111 & & \mathbf{1}01010
\end{array}$$

✓

3. Radixsort sorts an array A of length n consisting of d-digit numbers where each digit is from the set $\{0, 1, \dots, b\}$ in time $O(d(n + b))$.

   We are given an array A of n integers where each integer is *polynomially bounded*, i.e., each integer is from the range $\{0, 1, \dots, n^c\}$, for some constant c. Argue that Radixsort can be used to sort A in time $O(n)$.

   Hint: Find a suitable representation of the numbers in $\{0, 1, \dots, n^c\}$ as d-digit numbers where each digit comes from a set $\{0, 1, \dots, b\}$ so that Radixsort runs in time $O(n)$. How do you chose d and b?

Solution. We encode the numbers in A using digits from the set $\{0, 1, \dots, n-1\}$, i.e., we set $b = n-1$. To be able to encode all numbers in the range $\{0, 1, \dots, n^c\}$ it is required that $(b+1)^d \geq n^c + 1$ (we can encode $(b+1)^d$ different numbers using d digits where each digit comes from a set of cardinality $b+1$, and the cardinality of the set $\{0, 1, \dots, n^c\}$ is $n^c + 1$). Since $(b+1)^d = n^d$, we can set $d = c+1$, since

$$n^{c+1} \geq n^c + 1$$

holds for every $n \geq 2$ (assuming that $c \geq 1$). The runtime then is

$$O(d(n+b)) = O((c+1)(n+(n-1))) = O((c+1)2n) = O(n) ,$$

since 2 and $c+1$ are both constants. ✓

2 Recurrences: Substitution Method

1. Consider the following recurrence:

$$T(1) = 1 \text{ and } T(n) = T(n-1) + n$$

Show that $T(n) \in O(n^2)$ using the substitution method.

Solution. We need to show that $T(n) \leq C \cdot n^2$, for some suitable constant C. To this end, we first plug our guess into the recurrence:

$$T(n) = T(n-1) + n \leq C(n-1)^2 + n \ .$$

It is required that $C(n-1)^2 + n \leq Cn^2$:

$$\begin{aligned} C(n-1)^2 + n &\leq Cn^2 \\ C(n^2 - 2n + 1) + n &\leq Cn^2 \\ C - 2Cn + n &\leq 0 \\ C(1-2n) &\leq -n \\ C &\geq \frac{n}{2n-1} \ . \end{aligned}$$

Observe that $\frac{n}{2n-1} \leq 1$ holds for every $n \geq 1$. Our guess thus holds for every $C \geq 1$.

It remains to verify the base case. We have $T(1) = 1$ and $C1^2 = C$. Hence, $C1^2 \leq T(1)$ holds for every $C \geq 1$. We thus choose $C = 1$.

We have shown that $T(n) \leq Cn^2 = n^2$ holds for every $n \geq 1$. This implies that $T(n) = O(n^2)$. ✓

2. Consider the following recurrence:

$$T(1) = 1 \text{ and } T(n) = T(\lceil n/2 \rceil) + 1$$

Show that $T(n) \in O(\log n)$ using the substitution method.

Hint: Use the inequality $\lceil n/2 \rceil \leq \frac{n}{\sqrt{2}} = \frac{n}{2^{\frac{1}{2}}}$, which holds for all $n \geq 2$. Use $n = 2$ as your base case.

Solution. We need to show that $T(n) \leq C \cdot \log n$, for a suitable constant C. To this end, we plug our guess into the recurrence:

$$\begin{aligned} T(n) &= T(\lceil n/2 \rceil) + 1 \\ &\leq C \cdot \log\left(\lceil n/2 \rceil\right) + 1 \\ &\leq C \cdot \log\left(\frac{n}{\sqrt{2}}\right) + 1 \\ &= C\log(n) - C \cdot \frac{1}{2}\log(2) + 1 \\ &= C\log(n) - \frac{1}{2}C + 1 \ , \end{aligned}$$

where we used the inequality $\lceil n/2 \rceil \leq \frac{n}{\sqrt{2}}$. It is required that $C\log(n) - \frac{1}{2}C + 1 \leq C\log(n)$:

$$\begin{aligned} C\log(n) - \frac{1}{2}C + 1 &\leq C\log(n) \\ 1 &\leq \frac{1}{2}C \\ 2 &\leq C \ . \end{aligned}$$

The "induction step" part of the proof thus works for any $C \geq 2$. Regarding the base case, we will consider $n = 2$. We have:

$$T(2) = T(1) + 1 = 2 \ .$$

We thus need to show that $2 \leq C\log 2$. This holds for every $C \geq 2$. We can thus pick the value $C = 2$. This proves that $T(n) \in O(\log n)$. ✓

3 Search in a Sorted Matrix (Difficult!)

We are given an n-by-n integer matrix A that is sorted both row- and column-wise, i.e., every row is sorted in non-decreasing order from left to right, and every column is sorted in non-decreasing order from top to bottom. Give a divide-and-conquer algorithm that answers the question:

"Given an integer x, does A contain x?"

What is the runtime of your algorithm?

Solution. For simplicity, we assume that n is a power of two in this solution. We define the following submatrices of matrix A:

$$\begin{aligned} A_{11} &= A[0 \ldots \frac{n}{2} - 1, 0 \ldots \frac{n}{2} - 1] \\ A_{21} &= A[\frac{n}{2} \ldots n - 1, 0 \ldots \frac{n}{2} - 1] \\ A_{12} &= A[0 \ldots \frac{n}{2} - 1, \frac{n}{2} \ldots n - 1] \\ A_{22} &= A[\frac{n}{2} \ldots n - 1, \frac{n}{2} \ldots n - 1] \end{aligned}$$

Observe that the dimensions of all submatrices are $n/2 \times n/2$.

We first check whether $A_{\frac{n}{2}-1,\frac{n}{2}-1} = x$. If this is the case then we have found x and we are done. Otherwise, we distinguish the following two cases:

1. Suppose that $A_{\frac{n}{2}-1,\frac{n}{2}-1} < x$ holds. Then, since A is sorted in both column and row order, it is not hard to see that x is not contained in A_{11}. We then invoke our algorithm recursively and search for x in the three submatrices A_{12}, A_{21}, A_{22}.

2. Suppose that $A_{\frac{n}{2}-1,\frac{n}{2}-1} > x$ holds. Then, similar as before, it is not hard to see that x is not contained in A_{22}. We then invoke our algorithm recursively and search for x in the three submatrices A_{11}, A_{12}, A_{21}.

Observe that the proposed algorithm is a recursive algorithm. We thus need to decide what to do if the input to a recursive call is a 1×1 matrix. In this case we simply check whether the single element in the matrix equals x in $O(1)$ time.

Let $T(n)$ be the runtime of the algorithm when executed on an input array of dimension $n \times n$. We thus obtain the following recurrence:

$$T(n) = \begin{cases} O(1) \ , & \text{if } n = 1, \\ 3T(n/2) + O(1) \ , & \text{otherwise.} \end{cases}$$

It remains to solve the recurrence $T(n)$. First, we eliminate the $O(1)$ terms and replace them with a large enough constant C:

$$T(n) = \begin{cases} C \ , & \text{if } n = 1, \\ 3T(n/2) + C \ , & \text{otherwise.} \end{cases}$$

Our recursion is simple enough to obtain a solution via the recursion tree method. In the lecture, we used the recursion tree method in order to obtain a guess the we then verified using the substitution method. The recursion here is however simple enough to conduct a complete analysis using the recursion tree.

From the recursion tree, we see that the tree has $\log(n) + 1$ levels. Denoting the root of the tree as level 0, we see that level i has 3^i nodes. Furthermore, every node is labeled by C. The total work therefore is:

$$\begin{aligned} \sum_{i=0}^{\log n} 3^i C &= C \cdot \sum_{i=0}^{\log n} 3^i = C \cdot \frac{3^{\log(n)+1} - 1}{3 - 1} \\ &= \frac{C}{2} \cdot \left(2^{\log(3)\log(n)+\log(3)} - 1\right) \leq \frac{C}{2} \cdot \left(2^{\log(3)\log(n)+\log(3)}\right) \\ &= \frac{C}{2} \cdot \left(n^{\log 3} \cdot 3\right) = O(n^{\log 3}) \approx O(n^{1.5849\ldots}) \ . \end{aligned}$$

We used the formula $\sum_{i=0}^{k} x^i = \frac{x^{k+1}-1}{x-1}$ in this calculation.

Last, I would like to mention that there exists a solution to this problem that runs in time $O(n)$. Can you think of such a solution? ✓

4 Loop Invariant for Radixsort

Radixsort is defined as follows:

Require: Array A of length n consisting of d-digit numbers where each digit is taken from the set $\{0, 1, \ldots, b\}$

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for i = 1, ..., d do
    Use a stable sort algorithm to sort array A on digit i
end for
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(least significant digit is digit 1)

In this exercise we prove correctness of Radixsort via the following loop invariant:

At the beginning of iteration i of the for-loop, i.e., after i has been updated in Line 1 but Line 2 has not yet been executed, the following holds:

The integers in A are sorted with respect to their last $i-1$ digits.

1. *Initialization:* Argue that the loop-invariant holds for $i = 1$.

   Solution. In the beginning of the iteration with $i = 1$ the loop-invariant states that the integers in A are sorted with respect to their last $i - 1 = 0$ digits. This is trivially true. ✓

2. *Maintenance:* Suppose that the loop-invariant is true for some i. Show that it then also holds for $i + 1$.

   Hint: You need to use the fact that the employed sorting algorithm as a subroutine is stable.

   Solution. Suppose that the integers in A are sorted with respect to their last $i-1$ digits at the beginning of iteration i. We will show that at the beginning of iteration $i + 1$ the intergers are sorted with respect to their last i digits.

   Let A_{i+1} be the state of A in the beginning of iteration $i + 1$. For an integer x, let $x^{(i)}$ be the integer obtained by removing all but the last i digits from x. Suppose for the sake of a contradiction that there are indices j, k with $j < k$ such that $(A_{i+1}[j])^{(i)} > (A_{i+1}[k])^{(i)}$. If such integers exist then the loop invariant would not hold. We will show that assuming that these integers exist leads to a contradiction.

   First, suppose that digit i of $(A_{i+1}[j])^{(i)}$ and digit i of $(A_{i+1}[k])^{(i)}$ are identical. Note that this implies $(A_{i+1}[j])^{(i-1)} > (A_{i+1}[k])^{(i-1)}$. Observe that in iteration i, the digits are sorted with respect to digit i. Since the subroutine employed in Radixsort is a stable sort algorithm, the relative order of the two numbers has not changed since their ith digits are identical. This implies that the relative order of the two numbers was the same at the beginning of iteration i. This is a contradiction, since the loop invariant at the beginning of iteration i states that the digits are sorted with respect to their $i-1$ last digits, however, $(A_{i+1}[j])^{(i-1)} > (A_{i+1}[k])^{(i-1)}$ holds.

   Next, suppose that digit i of $(A_{i+1}[j])^{(i)}$ and digit i of $(A_{i+1}[k])^{(i)}$ are different. Then, since $(A_{i+1}[j])^{(i)} > (A_{i+1}[k])^{(i)}$ we have that digit i of $(A_{i+1}[j])^{(i)}$ is necessarily larger than digit i of $(A_{i+1}[k])^{(i)}$. This however is a contradiction to the fact that the numbers were sorted with respect to their ith digit in iteration i.

   Hence, the assumption that there are indices j, k such that $(A_{i+1}[j])^{(i)} > (A_{i+1}[k])^{(i)}$ is wrong. If no such indices exist then the integers in A are sorted with respect to their last i digits at the beginning of iteration $i + 1$. ✓

3. *Termination:* Use the loop-invariant to conclude that A is sorted after the execution of the algorithm.

   Solution. After iteration d (or before iteration $d + 1$, which is never executed), the invariant states that the numbers in A are sorted with respect to their last d digits, which simply means that all numbers are now sorted with regards to all their digits. ✓