

Big- O Notation

COMS10017 - (Object-Oriented Programming and) Algorithms

Dr Christian Konrad

Definition: O -notation (“Big O”)

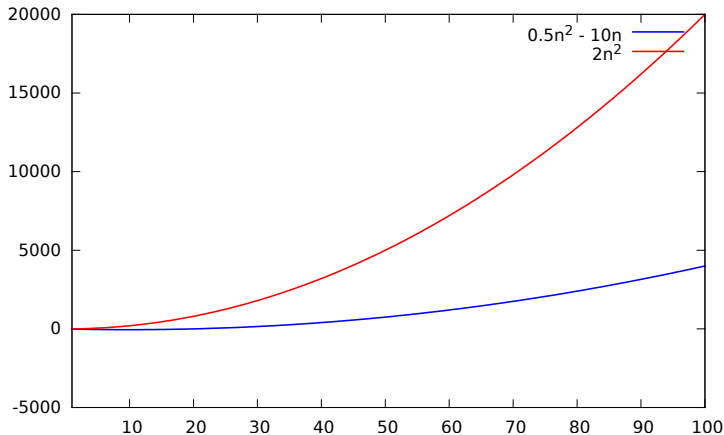
Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be a function. Then $O(g(n))$ is the set of functions:

$$O(g(n)) = \{f(n) : \text{There exist positive constants } c \text{ and } n_0 \text{ such that } 0 \leq f(n) \leq cg(n) \text{ for all } n \geq n_0\}$$

Meaning: $f(n) \in O(g(n))$: “ g grows asymptotically at least as fast as f up to constants”

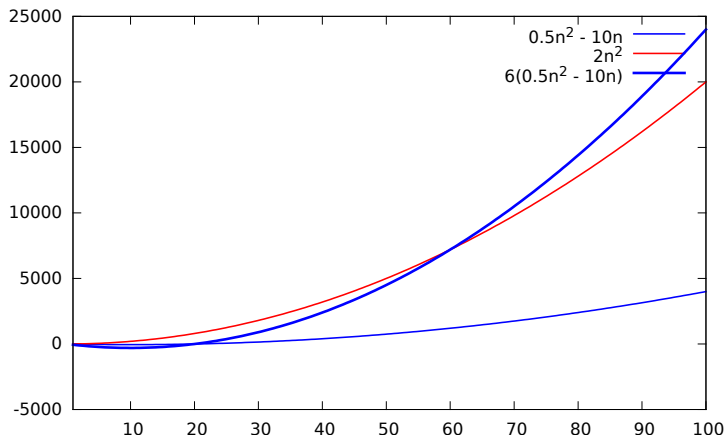
O-Notation: Example

Example: $f(n) = \frac{1}{2}n^2 - 10n$ and $g(n) = 2n^2$



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Then: $g(n) \in O(f(n))$, since $6f(n) \geq g(n)$, for every $n \geq n_0 = 60$

More Examples/Exercises

Recall:

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Exercises:

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$$\log n \leq 2c$$

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$$\begin{aligned} 0.5n &\leq cn/\log n \\ \log n &\leq 2c \\ n &\leq 2^{2c}, \text{ a contradiction,} \end{aligned}$$

since this does not hold for every $n > 2^{2c}$.



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Proving that $f \notin O(g)$:

Proof by contradiction: Assume that constants c, n_0 exist as in the statement of the definition of Big- O and derive a contradiction

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Lemma (Sum of Two Functions)

Suppose that $f, g \in O(h)$. Then: $f + g \in O(h)$.

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$$f(n) + g(n) \leq ch(n) + c'h(n) = C \cdot h(n) \text{ for every } n \geq N_0 . \quad \square$$

Further Properties

Lemma (Polynomials)

Let $f(n) = c_0 + c_1n + c_2n^2 + c_3n^3 + \dots + c_kn^k$, for some integer k that is independent of n . Then: $f(n) \in O(n^k)$.

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Application of statement on last slide n times! (only allowed to apply statement $O(1)$ times!)

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Important Properties for the Analysis of Algorithms

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- Loops: (repetition of instructions)

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- Super-exponential time: e.g. $O(2^{2^n})$ (big trouble...)