Linear and Binary Search

COMS10017 - (Object-Oriented Programming and) Algorithms

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Runtime of Algorithms

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Consider an algorithm $\mathcal{A}$ for a specific problem $\mathcal{P}$

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Worst-case Runtime: $\max_{X \in S(n)} T(X)$

Best-case Runtime: $\min_{X \in S(n)} T(X)$

Average-case Runtime: $\frac{1}{|S(n)|} \sum_{X \in S(n)} T(X)$

Linear Search

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- **Input:** Array A of n integers from range $\{0, 1, 2, \ldots, k-1\}$, for some integer k, integer $t \in \{0, 1, 2, \ldots, k-1\}$
- **Output:** 1, if A contains t, 0 otherwise

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Best-case Runtime: $O(1)$
On any input with $A[0] = t$

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Require: Array A, integer t
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Average-case Runtime: (over all possible inputs of length n)

Average-case Analysis of Linear Search

Possible Inputs of Length n

$$S(n) := \{\text{arrays } A \text{ of length } n \text{ with } A[i] \in \{0, 1, 2, \dots, k-1\}, \text{ for every } 0 \leq i \leq n-1\}$$

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Auxiliary Function: For $A \in S(n), t \in \{0, 1, \dots, k-1\}$:

$$\textsc{Left}(A, t) = \min\{i \ : \ A[i] = t\} \ .$$

If no such position exists then $\textsc{Left}(A, t) = n$.

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$\rightarrow$ Linear search loop executed $\textsc{Left}(X,t) + 1$ times

Average-case Analysis of Linear Search (continued)

Average-case Runtime for $k = 2$: (binary strings)

We compute average number of steps the loop is executed $(t = 1)$

$$\mathsf{AVG} = \frac{1}{|S(n)|} \sum_{A \in S(n)} \textsc{Left}(A, 1) + 1$$

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$$\begin{aligned}
\text{AVG} &= \frac{1}{|S(n)|} \sum_{A \in S(n)} \textsc{Left}(A, 1) + 1 \\
&= 2^{-n} \left(\left(\sum_{i=0}^{n-1} |\{A : \textsc{Left}(A, 1) = i\}| \cdot (i + 1) \right) + (n + 1) \right) .
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\textcolor{blue}{\underbrace{0\ 0\ 0\ 0\ \ldots 0}_{i \text{ times}}}\ \textcolor{blue}{1}\ \textcolor{red}{\underbrace{X\ X\ X\ \ldots\ X}_{n-i-1 \text{ times}}}
$$

$$
= 2^{-n} \left(\left(\sum_{i=0}^{n-1} 2^{n-1-i} \cdot (i+1) \right) + (n+1) \right) \rightarrow
$$

AVG-case runtime is $O(1)$

$$
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How to bound $\sum_{i=0}^{n} \frac{i}{2^i}$:

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&= \left(\sum_{i=1}^{n} \frac{1}{2^i} \right) - \frac{n}{2^{n+1}} \leq \frac{\frac{1}{2} - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} \leq 1 \ .
\end{aligned}$$

$\rightarrow S_n \leq 2$

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```
Require: Sorted array A of length n, integer t
  if |A| ≤ 2 then
    Check A[0] and A[1] and return answer
  if A[⌊n/2⌋] = t then
    return ⌊n/2⌋
  else if A[⌊n/2⌋] > t then
    return BINARY-SEARCH(A[0, ..., ⌊n/2⌋ − 1])
  else
    return ⌊n/2⌋ + 1 + BINARY-SEARCH(A[⌊n/2⌋ + 1, n − 1])
```

Algorithm BINARY-SEARCH

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Worst-case runtime of Binary Search: $O(\log n)$