

Proofs by Induction (Recap)

COMS10017 - (Object-Oriented Programming and) Algorithms

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- This is often done by induction
- We will use proofs by induction for proving loop invariants (soon) and for solving recurrences (later)

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- 1 **Statement to Prove:**
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④ **Base case:** Prove that $P(1)$ holds

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Example: $a^n = 1$, for every $a \neq 0$ and n nonnegative integer

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- 2 Induction hypothesis: $a^m = 1$, for every $0 \leq m \leq n$ (strong induction)
- 3 Induction step:

$$a^{n+1} = a^{2n-(n-1)} = \frac{a^{2n}}{a^{n-1}} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1 \dots$$

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Problem: a^1 is computed as $\frac{a^0 a^0}{a^{-1}}$ and induction hypothesis does not hold for $n = -1$!