Runtime of Quicksort

COMS10017 - (Object-Oriented Programming and) Algorithms

Dr Christian Konrad

Quicksort

Require: array A of length n
  if $n \leq 1$ **then**
    return A
  else
    $i \leftarrow$ Partition(A)
    $\textsc{Quicksort}(A[0, i-1])$
    $\textsc{Quicksort}(A[i+1, n-1])$

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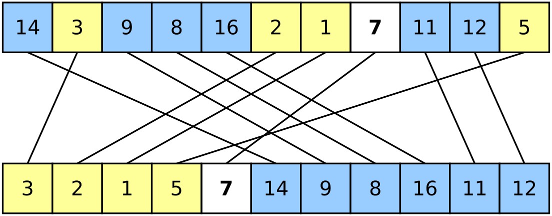

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Partition: Let C be such that Partition() runs in time at most Cn

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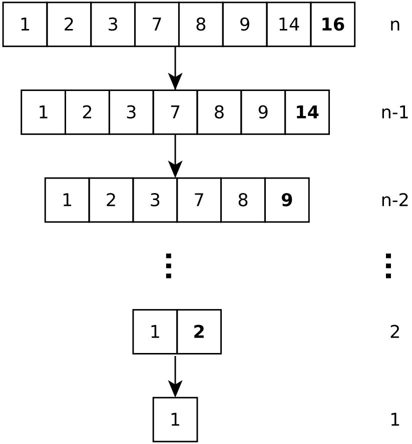

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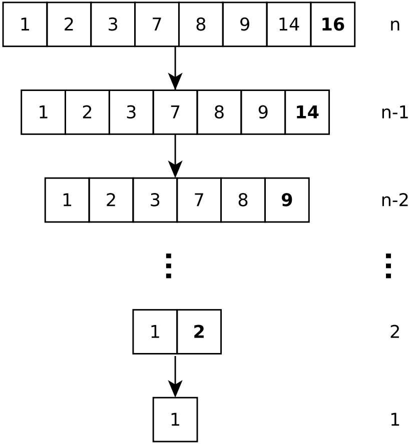

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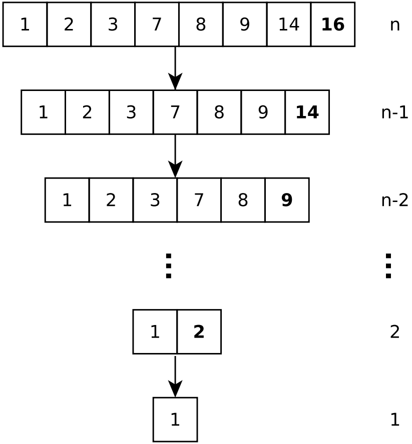

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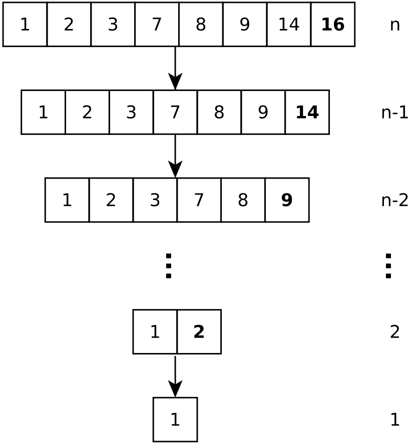

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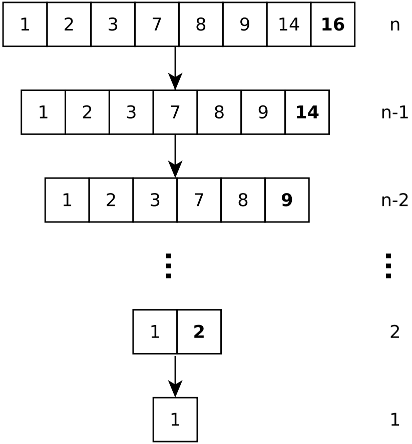

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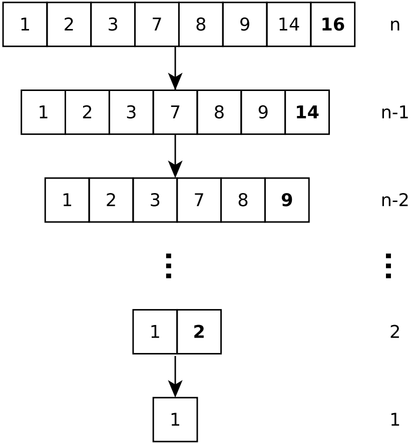

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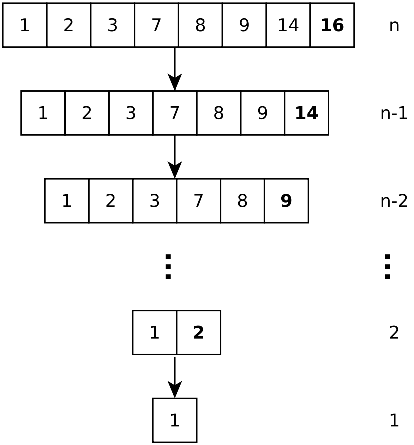

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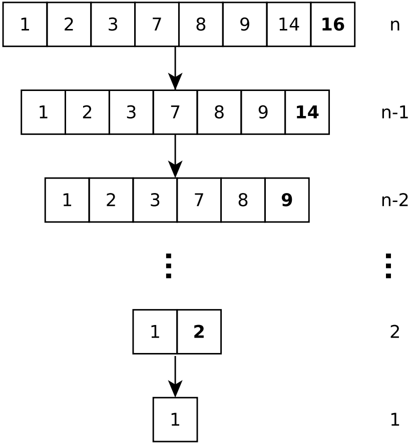

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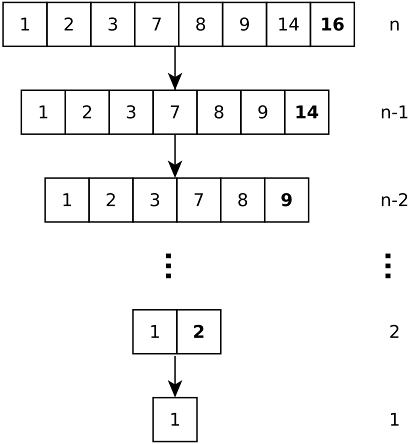

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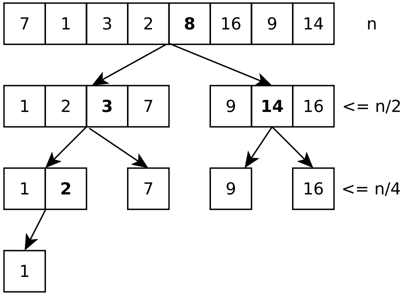

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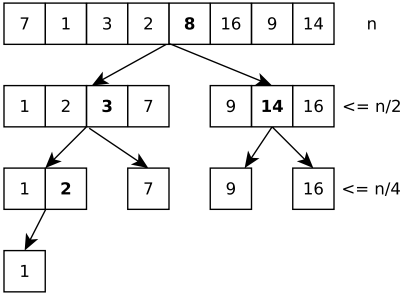

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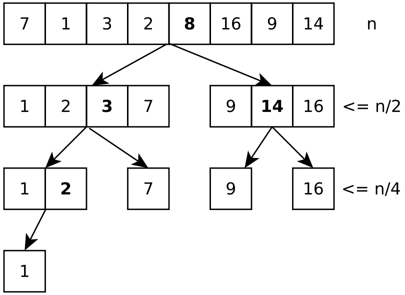

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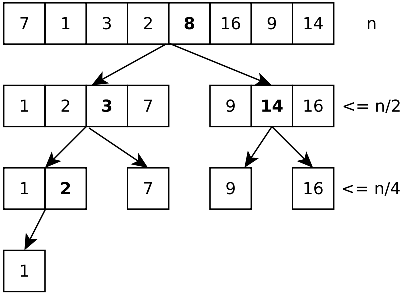

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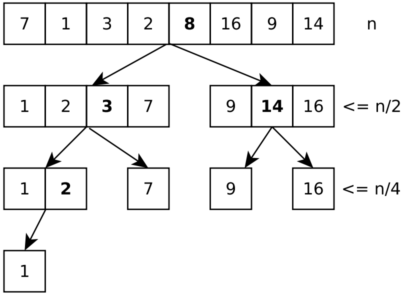

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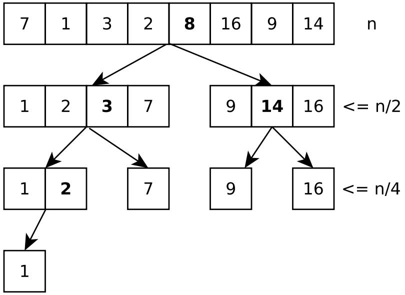

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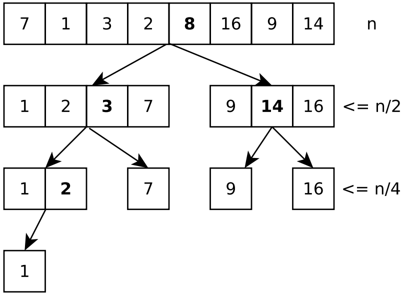

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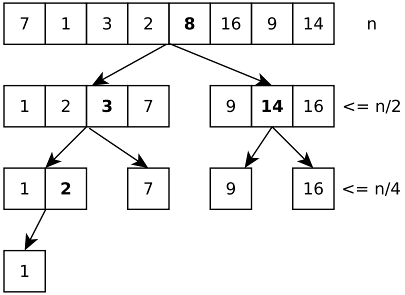

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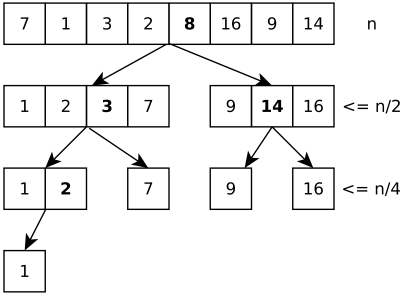

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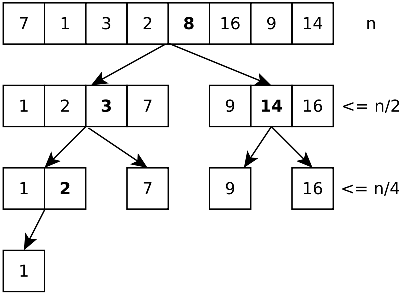

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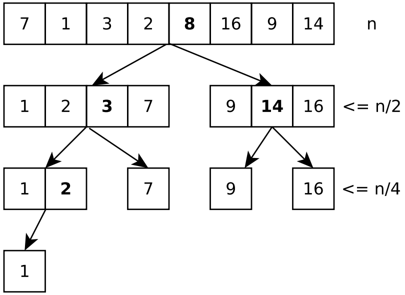

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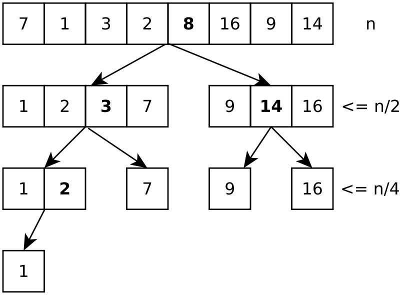

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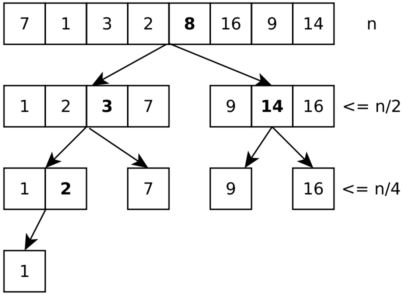

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- Total runtime: $\ell \cdot O(n) = O(n \log n)$.

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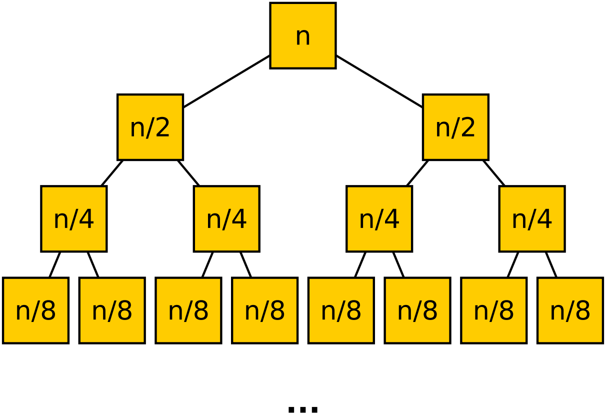
Good versus Bad Splits: Intuition and Rough Analysis



Only good splits: Recursion tree depth $\lceil \log n \rceil + 1$

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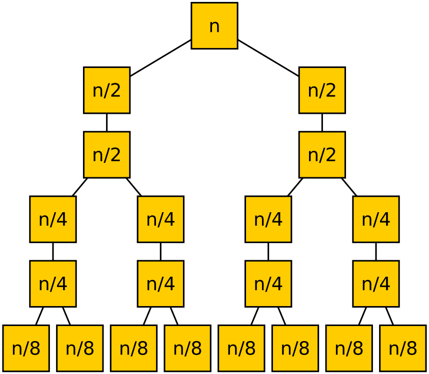

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Good & bad splits alternate: Recursion tree depth $2 \cdot (\lceil \log n \rceil + 1)$

Selecting good Pivots

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Idea that works in Practice: Select Pivot at random!
(Implementation: exchange $A[n-1]$ with a uniform random element $A[i]$)

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Definition (Bad Split)
A split is *bad* if $\min\{n_1, n_2\} \leq \frac{1}{10} n$.

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- *Expected runtime*: Since we introduce randomness, the runtime of the algorithm becomes a random variable

Definition (Bad Split)
A split is *bad* if $\min\{n_1, n_2\} \leq \frac{1}{10} n$.

If we select the pivot randomly, how likely is it to have a bad split?

Probability of a Bad Split

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Random Pivot Selection: QUICKSORT runs in expected time $O(n \log n)$ if the pivot is chosen uniformly at random