The Fibonacci Numbers

COMS10017 - (Object-Oriented Programming and) Algorithms

Dr Christian Konrad

The Fibonacci Numbers

Fibonacci Numbers

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F_0 &= 0 \\
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F_n &= F_{n-1} + F_{n-2} \text{ for } n \geq 2 \ .
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Why are they important?

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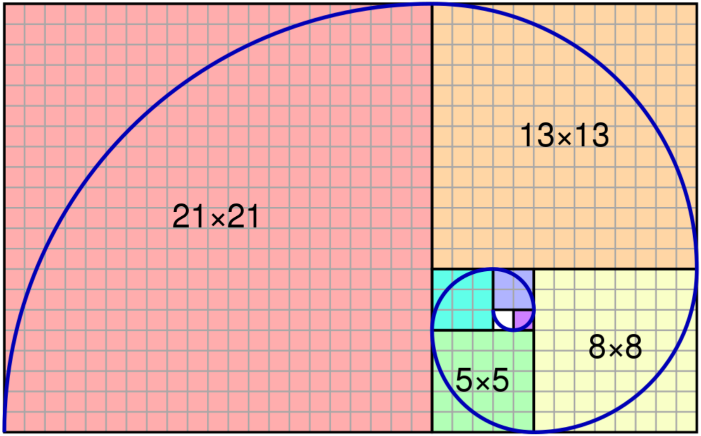
Why are they important?

- Fibonacci heaps (data structure)
- Appear in analysis of algorithms (e.g. Euclid's algorithm)
- Appear everywhere in nature
- Interesting and instructive computational problem



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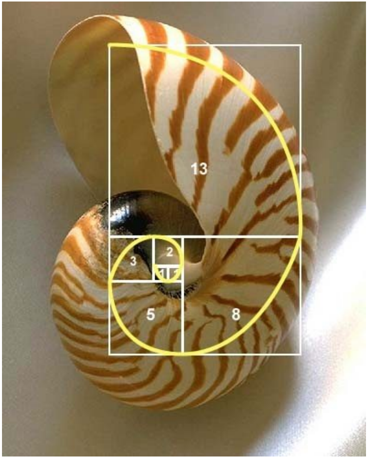

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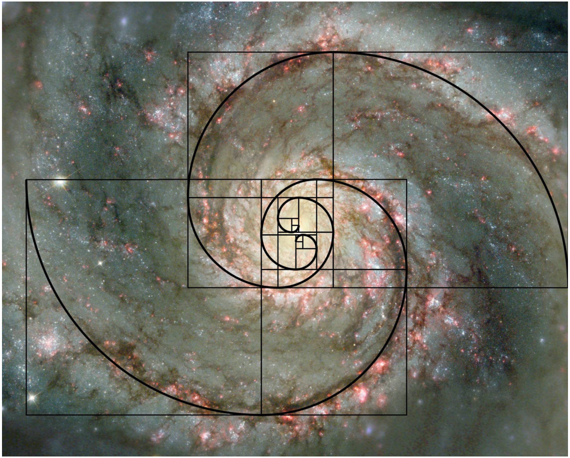

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Computing the Fibonacci Numbers

Naïve Algorithm

Require: Integer $n \geq 0$
  if $n \leq 1$ **then**
    return n
  else
    return $\textsc{Fib}(n-1) + \textsc{Fib}(n-2)$

$\textsc{Fib}(n)$

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Computing the Fibonacci Numbers

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- Without recursive calls, runtime is $O(1)$
- Hence, runtime is $O($"number of recursive calls"$)$

Runtime Analysis

Define Recurrence:

$T(n)$: number of recursive calls to $\textsc{Fib}$ when called with parameter n

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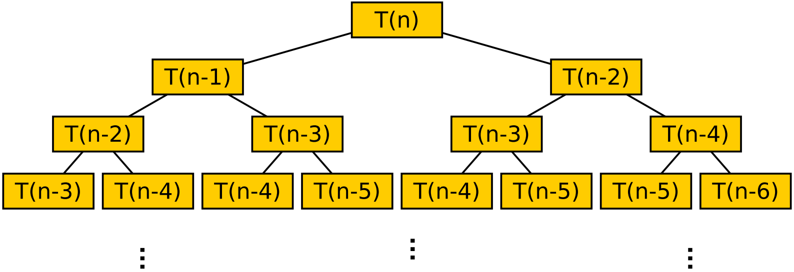
How to Solve this Recurrence?

- We will use the recursion tree technique to obtain a guess for an upper bound
- We will verify the guess with the substitution method

Recursion Tree for T



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- Each node contributes 1
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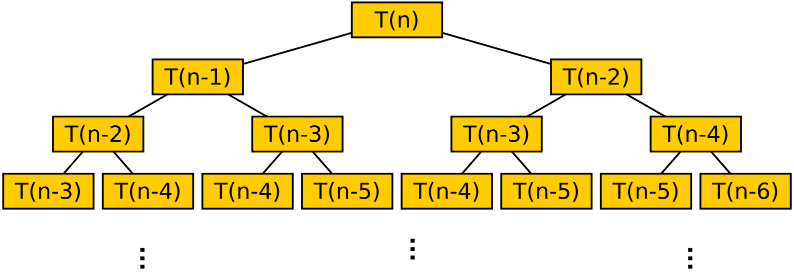

Observe:

- Each node contributes 1
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- Number of levels of recursion tree: n
- Our guess: $T(n) \leq c^n$ (we believe $c \leq 2$)

Verification with the Substitution Method

Recall:

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- The additive 1 prevents us from getting a similar form as c^n
- Try different guess: $T(n) \leq c^n - 1$

Verification with the Substitution Method (2)

New Guess: $T(n) \leq c^n - 1$

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- $c^0 - 1 = 0$ and $c^1 - 1 \approx 0.61$ ✗

Verification with the Substitution Method (3)

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We proved $T(n) \leq 2 \cdot (\frac{1+\sqrt{5}}{2})^n - 1$. Hence $T(n) \in O\left((\frac{1+\sqrt{5}}{2})^n\right)$.

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Golden Ratio:

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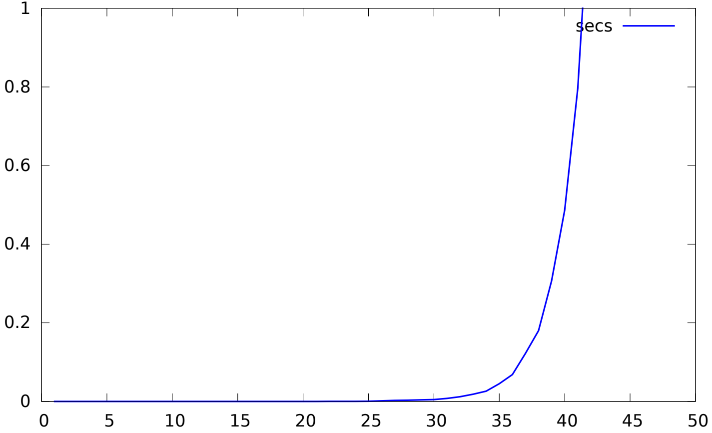
- Large numbers involved
- Impractical

Experiments



Exponential growth

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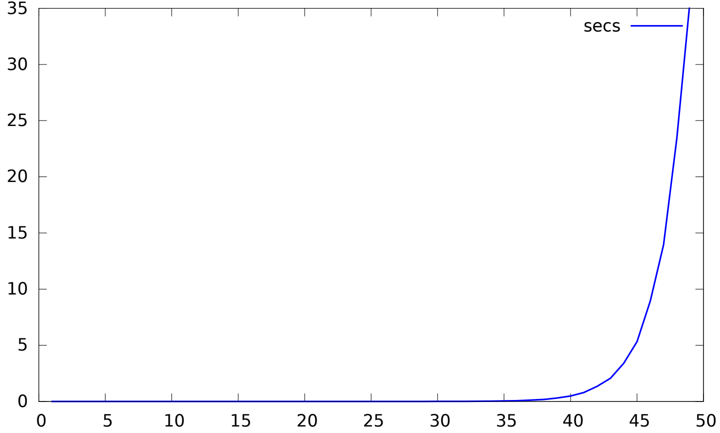

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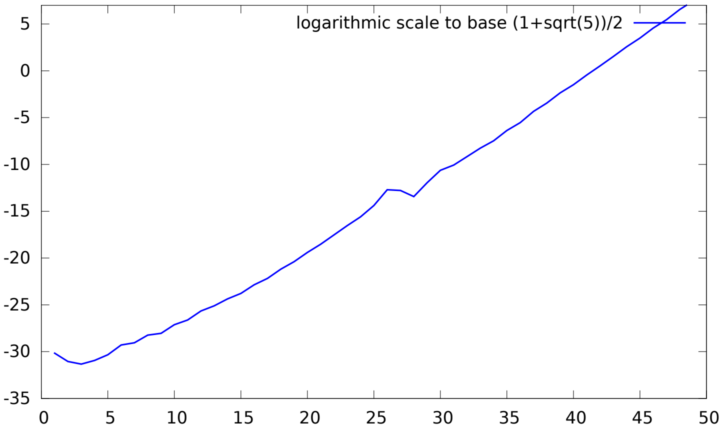
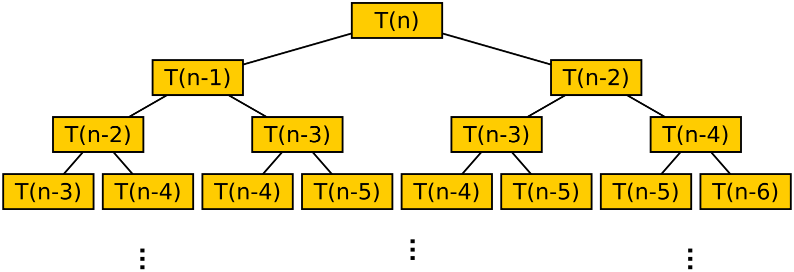


Logarithmic scale

Why is this Algorithm so slow?



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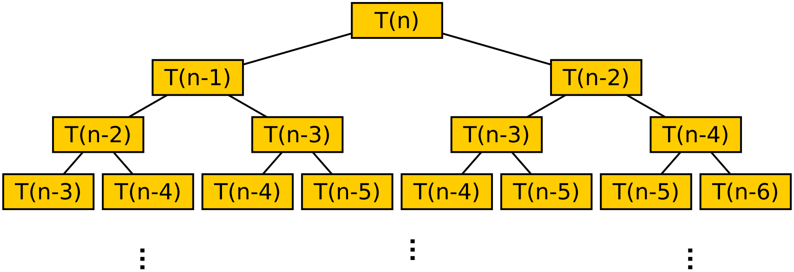

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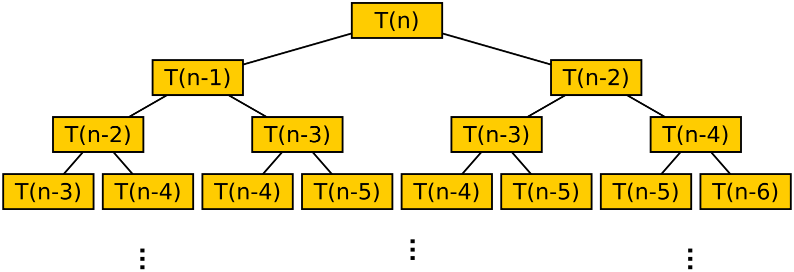

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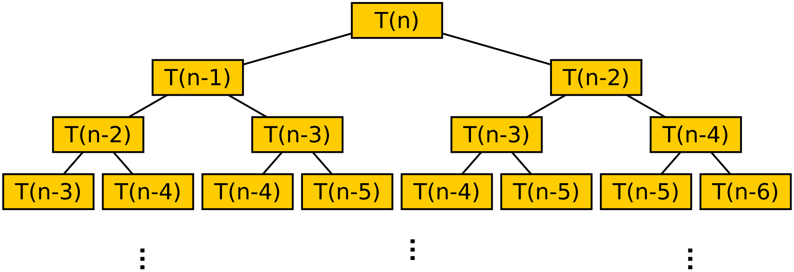

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Dynamic Programming!

Dynamic Programming Solution

Dynamic Programming (will be discussed in more detail later)

- Store solutions to subproblems in a table
- Compute table bottom up

Require: Integer $n \geq 0$
  if $n \leq 1$ **then**
    return n
  else
    $A \leftarrow$ array of size n
    $A[0] \leftarrow 1$, $A[1] \leftarrow 1$
    for $i \leftarrow 2 \ldots n$ **do**
      $A[i] \leftarrow A[i-2] + A[i-1]$
    return $A[n]$

DynPrgFib(n)

Dicussion

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- DynPrgFib() runs in time $O(n)$
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Can we reduce the space to $O(1)$?

Improvement:

- Observe that when $T(i)$ is computed, the values $T(1), T(2), \dots, T(i-3)$ are no longer needed
- Only store the last two values of T

Improved Algorithm

```
Require: Integer n ≥ 0
  if n ≤ 1 then
    return n
  else
    a ← 0
    b ← 1
    for i ← 2 ... n do
      c ← a + b
      a ← b
      b ← c
    return c
```

$\textsc{ImprovedDynPrgFib}(n)$

Correctness: via loop invariant!