Exercise Sheet 3
COMS10017 Algorithms 2022/2023

Reminder: $\log n$ denotes the binary logarithm, i.e., $\log n = \log_2 n$.

Example Question: Loop Invariants

Question. Prove that the stated invariant holds throughout the execution of the loop (using the Initialization, Maintenance, Termination approach discussed in the lectures):

Algorithm 1

Require: Array A of length n ($n \geq 2$)
1: $S \leftarrow A[0] - A[1]$
2: **for** $i \leftarrow 1 \dots n-2$ **do**
3: $\quad S \leftarrow S + A[i] - A[i+1]$
4: **end for**
5: **return** S

Invariant:

At the beginning of iteration i, the statement $S = A[0] - A[i]$ holds.

Which value is returned by the algorithm (use the Terminiation property for this)?

Solution. Let S_i be the value of S at the beginning of iteration i.

1. *Initialization* ($i = 1$): We need to show that the statement of the loop invariant holds for $i = 1$, i.e., the statement $S_1 = A[0] - A[1]$ holds before iteration $i = 1$. Observe that, in Line 1, S_1 is initialized as $S_1 \leftarrow A[0] - A[1]$. The loop invariant thus holds for $i = 1$.

2. *Maintenance*: Assume that the loop invariant holds for value i, i.e., $S_i = A[0] - A[i]$. We need to show that the loop invariant then also holds for value $i + 1$, i.e., we need to show that $S_{i+1} = A[0] - A[i + 1]$ holds. To this end, observe that in iteration i we execute the operation $S_{i+1} = S_i + A[i] - A[i + 1]$. Since $S_i = A[0] - A[i]$, we obtain $S_{i+1} = A[0] - A[i] + A[i] - A[i + 1] = A[0] - A[i + 1]$.

3. *Termination*: We have that, after the last iteration (or before the $(n-1)$th iteration that is never executed), $S_{n-1} = A[0] - A[n - 1]$ holds. The algorithm thus returns the value $A[0] - A[n - 1]$.

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1 Warm up: Proof by Induction

Consider the following sequence: $s_1 = 1, s_2 = 2, s_3 = 3$, and $s_n = s_{n-1} + s_{n-2} + s_{n-3}$, for every $n \geq 4$. Prove that the following holds:

$$s_n \leq 2^n \ .$$

2 Loop Invariant

Prove that the stated loop invariant holds throughout the execution of the loop (using the Initialization, Maintenance, Termination approach discussed in the lectures):

Algorithm 2

Require: Array A of n positive integers

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B ← empty array of n integers
B[0] ← A[0]
for i = 1 ... n − 1 do
    if A[i] > B[i − 1] then
        B[i] ← A[i]
    else
        B[i] ← B[i − 1]
    end if
end for
return B[n − 1]
```

Loop Invariant: At the beginning of iteration i, the following statement holds: For every $0 \leq j < i$: $B[j]$ is the maximum of the subarray $A[0, j]$, i.e., $B[j] = \max\{A[0], \ldots, A[j]\}$.

Which value is returned by the algorithm (use the Terminiation property for this)?

Hint: The Maintenance part requires a case distinction in order to deal with the if-else statement.

3 Insertionsort

What is the runtime (in Θ-notation) of Insertionsort when executed on the following arrays of lengths n:

1. $1, 2, 3, 4, \ldots, n-1, n$
2. $n, n-1, n-2, \ldots, 2, 1$
3. The array A such that $A[i] = 1$ if $i \in \{1, 2, 4, 8, 16, \ldots\}$ (i.e., when i is a power of two) and $A[i] = i$ otherwise.
4. The array B such that $B[i] = 1$ if $i \in \{10, 20, 30, 40 \ldots\}$ (i.e., when i is a multiple of 10) and $B[i] = i$ otherwise.
5. The array C such that $C[i] = 1$ if $i \in \{n^{\frac{1}{10}}, 2 \cdot n^{\frac{1}{10}}, 3 \cdot n^{\frac{1}{10}}, \ldots\}$ (i.e., when i is a multiple of $n^{\frac{1}{10}}$) and $C[i] = i$ otherwise. We assume here that $n^{\frac{1}{10}}$ is an integer.

4 Runtime Analysis

Algorithm 3

Require: Integer $n \geq 2$
  $x \leftarrow 0$
  $i \leftarrow n$
  while $i \geq 2$ **do**
    $j \leftarrow \lceil n^{1/4} \rceil \cdot i$
    while $j \geq i$ **do**
      $x \leftarrow x + 1$
      $j \leftarrow j - 10$
    end while
    $i \leftarrow \lfloor i/\sqrt{n} \rfloor$
  end while
  return x

Determine the runtime of Algorithm 3 in Θ-notation.

5 Optional and Difficult Questions

Exercises in this section are intentionally more difficult and are there to challenge yourself.

5.1 Proof by Induction

Let n be a positive number that is divisible by 23, i.e., $n = k \cdot 23$, for some interger $k \geq 1$. Let $x = \lfloor n/10 \rfloor$ and let $y = n \ \% \ 10$ (the rest of an integer division). Prove by induction on k that 23 divides $x + 7y$.

Example: Consider $k = 4$. Then $n = 92$, $x = 9$ and $y = 2$. Observe that the quantity $x + 7y = 9 + 7 \cdot 2 = 23$ is divisible by 23.