Peak Finding

COMS10017 - Algorithms 1

Dr Christian Konrad

Peak Finding

Let $A = a_0, a_1, \ldots, a_{n-1}$ be an array of integers of length n

0	1	2	3	4	5	6	7	8	9
a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9

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Definition: (Peak)
Integer a_i is a *peak* if adjacent integers are not larger than a_i

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4	3	9	10	14	8	7	2	2	2

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Peak Finding: Simple Algorithm

Problem Peak Finding: Write algorithm with properties:

1. **Input:** An integer array of length n
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```
int peak(int *A, int len) {
    if(A[0] >= A[1])
        return 0;
    if(A[len-1] >= A[len-2])
        return len-1;

    for(int i=1; i < len-1; i=i+1) {
        if(A[i] >= A[i-1] && A[i] >= A[i+1])
            return i;
    }
    return -1;
}
```

C++ code

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Require: Integer array A of length n
  if A[0] ≥ A[1] then
    return 0
  if A[n − 1] ≥ A[n − 2] then
    return n − 1
  for i = 1 ... n − 2 do
    if A[i] ≥ A[i − 1] and A[i] ≥ A[i + 1] then
      return i
  return −1
```

Pseudo code

Peak Finding: Problem well-defined?

Is Peak Finding well defined? Does every array have a peak?

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Every integer array has at least one peak.

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Proof.
Let A be an integer array of length n. Suppose for the sake of a contradiction that A does not have a peak. Then $a_1 > a_0$ since otherwise a_0 is a peak. But then $a_2 > a_1$ since otherwise a_1 is a peak. Continuing, for the same reason, $a_i > a_{i-1}$ since otherwise a_{i-1} is a peak, for every $i \leq n-1$. But this implies $a_{n-1} > a_{n-2}$ and hence a_{n-1} is a peak. A contradiction. Hence, every array has a peak. □

0	1	2	3	4	5	6
a_0	a_1	a_2	a_3	a_4	a_5	a_6

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a_0	$> a_0$	a_2	a_3	a_4	a_5	a_6

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a_0	$> a_0$	$> a_1$	$> a_2$	$> a_3$	$> a_4$	$> a_5$

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Proof.

Every maximum is a peak. (Shorter and immediately convincing!) □

Peak Finding: How fast is the Simple Algorithm?

How fast is our Algorithm?

```
Require: Integer array A of length n
  if A[0] ≥ A[1] then
    return 0
  if A[n − 1] ≥ A[n − 2] then
    return n − 1
  for i = 1 ... n − 2 do
    if A[i] ≥ A[i − 1] and A[i] ≥ A[i + 1] then
      return i
  return −1
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How often do we look at the array elements? (worst case!)

- $A[0]$ and $A[n-1]$:

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- $A[0]$ and $A[n-1]$: twice

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> **Require:** Integer array A of length n
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>   **return** 0
>   **if** $A[n-1] \geq A[n-2]$ **then**
>   **return** $n-1$
>   **for** $i = 1 \ldots n-2$ **do**
>   **if** $A[i] \geq A[i-1]$ **and** $A[i] \geq A[i+1]$ **then**
>   **return** i
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- $A[0]$ and $A[n-1]$: twice
- $A[1] \ldots A[n-2]$: 4 times (at most)

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  for i = 1 ... n − 2 do
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```

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- $A[0]$ and $A[n-1]$: twice
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- Overall: $2 + 2 + (n-2) \cdot 4 = 4(n-1)$

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  for i = 1 ... n − 2 do
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  return −1
```

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- $A[0]$ and $A[n-1]$: twice
- $A[1] \ldots A[n-2]$: 4 times (at most)
- Overall: $2 + 2 + (n-2) \cdot 4 = 4(n-1)$

Can we do better?!

Peak Finding: An even faster Algorithm

Finding Peaks even Faster: FAST-PEAK-FINDING

1. **if** A is of length 1 **then return** 0
2. **if** A is of length 2 **then** compare $A[0]$ and $A[1]$ and **return** position of larger element
3. **if** $A[\lfloor n/2 \rfloor]$ is a peak **then return** $\lfloor n/2 \rfloor$
4. Otherwise, **if** $A[\lfloor n/2 \rfloor - 1] \geq A[\lfloor n/2 \rfloor]$ **then**
   return FAST-PEAK-FINDING($A[0, \lfloor n/2 \rfloor - 1]$)
5. **else**
   return $\lfloor n/2 \rfloor + 1 +$
   FAST-PEAK-FINDING($A[\lfloor n/2 \rfloor + 1, n - 1]$)

Comments:

- FAST-PEAK-FINDING is *recursive* (it calls itself)
- $\lfloor x \rfloor$ is the floor function ($\lceil x \rceil$: ceiling)

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

Check whether $A[\lfloor n/2 \rfloor] = A[\lfloor 16/2 \rfloor] = A[8]$ is a peak

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

If $A[7] \geq A[8]$ then **return** $\textsc{Fast-Peak-Finding}(A[0,7])$

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

Length of subarray is 8

Peak Finding: Example

Example:

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3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

Check whether $A[\lfloor n/2 \rfloor] = A[\lfloor 8/2 \rfloor] = A[4]$ is a peak

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

If $A[3] \geq A[4]$ then **return** $\text{Fast-Peak-Finding}(A[0,3])$

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

Length of subarray is 4

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

Check whether $A[\lfloor n/2 \rfloor] = A[\lfloor 4/2 \rfloor] = A[2]$ is a peak

Peak Finding: Example

Example:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

If $A[1] \geq A[2]$ then **return** $\textsc{Fast-Peak-Finding}(A[0,1])$

Peak Finding: Example

Example:

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3	7	22	47	36	33	31	30	25	21	20	15	7	4	10	22

Else **return** FAST-PEAK-FINDING($A[3]$), which returns 3

Peak Finding: How fast is the Improved Algorithm?

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- Let $R(n)$ be the number of calls to Fast-Peak-Finding when the input array is of length n. Then:

$$R(1) \;=\; R(2) = 1$$

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$$\begin{aligned} R(1) &= R(2) = 1 \\ R(n) &\leq R(\lfloor n/2 \rfloor) + 1 \text{ , for } n \geq 3 \text{ .} \end{aligned}$$

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- Solving the recurrence (see lecture on recurrences):

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$$\begin{aligned} R(n) &\leq R(\lfloor n/2 \rfloor) + 1 \leq R(n/2) + 1 = R(\lfloor n/4 \rfloor) + 2 \\ &\leq R(n/4) + 2 = \dots \leq \lceil \log n \rceil \text{ .} \end{aligned}$$

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- Without the recursive calls, the algorithm looks at the array elements at most 5 times
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- Hence, we look at most at $5\lceil \log n \rceil$ array elements!

Peak Finding: Correctness

Why is the Algorithm correct?!

Steps 1,2,3 are clearly correct

1. **if** A is of length 1 **then return** 0
2. **if** A is of length 2 **then** compare $A[0]$ and $A[1]$ and **return** position of larger element
3. **if** $A[\lfloor n/2 \rfloor]$ is a peak **then return** $\lfloor n/2 \rfloor$
4. Otherwise, **if** $A[\lfloor n/2 \rfloor - 1] \geq A[\lfloor n/2 \rfloor]$ **then**
   return $\textsc{Fast-Peak-Finding}(A[0, \lfloor n/2 \rfloor - 1])$
5. **else**
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Why is step 4 correct? (step 5 is similar)

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Why is step 4 correct? (step 5 is similar)

- Need to prove: peak in $A[0, \lfloor n/2 \rfloor - 1]$ is a peak in A
- This is trivially true for every position $i < \lfloor n/2 \rfloor - 1$, since both cells adjacent to $A[i]$ are also contained in $A[0, \lfloor n/2 \rfloor - 1]$
- **Critical case:** $\lfloor n/2 \rfloor - 1$ is a peak in $A[0, \lfloor n/2 \rfloor - 1]$

Peak Finding: Correctness (2)

Why is the Algorithm correct?!

Steps 1,2,3 are clearly correct

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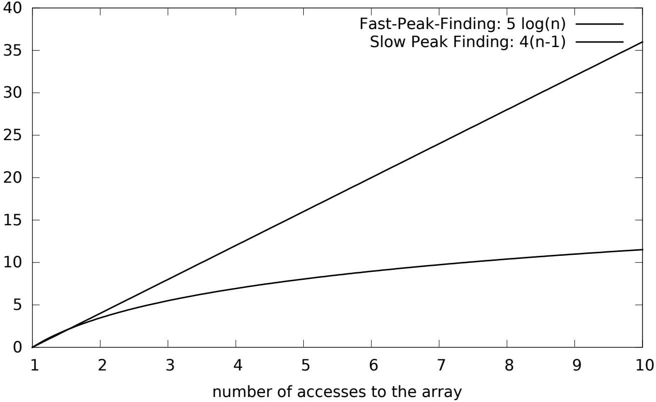
5. **else**
   return $\lfloor n/2 \rfloor + 1 +$ $\textsc{Fast-Peak-Finding}(A[\lfloor n/2 \rfloor + 1, n - 1])$

- **Critical case:** $\lfloor n/2 \rfloor - 1$ is a peak in $A[0, \lfloor n/2 \rfloor - 1]$
- Need to guarantee that $A[\lfloor n/2 \rfloor] \leq A[\lfloor n/2 \rfloor - 1]$ since otherwise $\lfloor n/2 \rfloor - 1$ would not be a peak
- This, however, follows from the condition in step 4! □

Peak Finding: Runtime Comparison

$4(n-1)$ **versus** $5 \log n$



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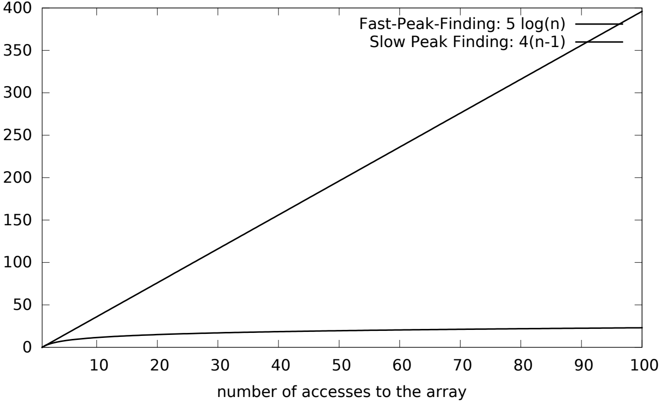

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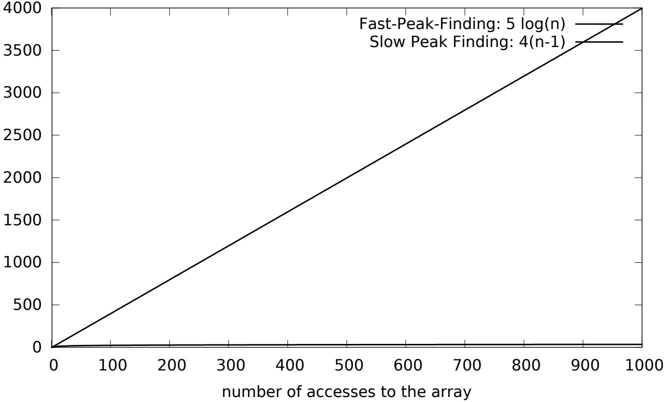

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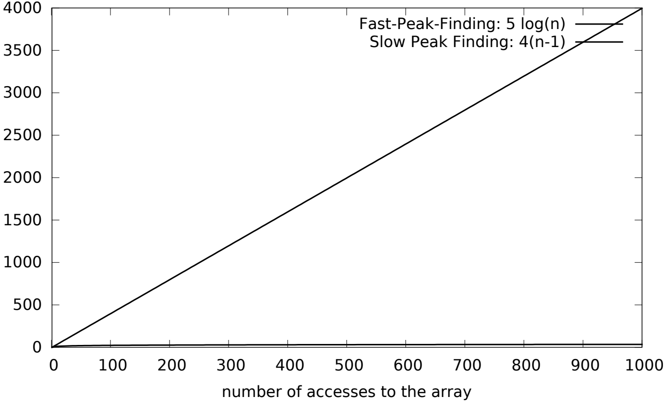

Conclusion: $5 \log n$ is so much better than $4(n-1)$!