Merge-sort

COMS10017 - Algorithms 1

Dr Christian Konrad

Definition of the Sorting Problem

Sorting Problem

- **Input:** An array A of n numbers
- **Output:** A reordering of A s.t. $A[0] \leq A[1] \leq \cdots \leq A[n-1]$

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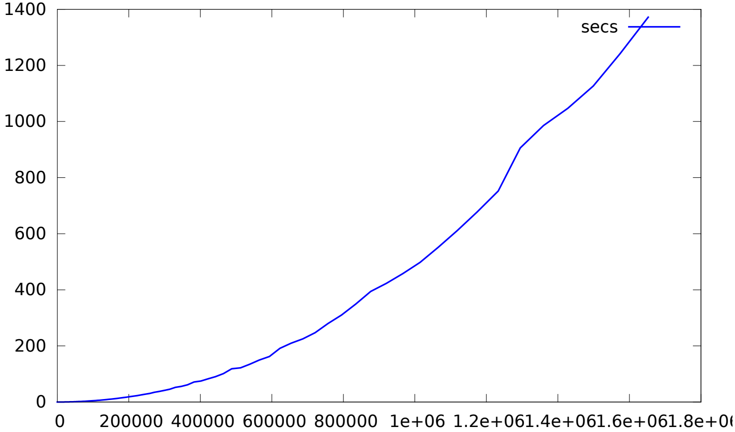
Insertion Sort

- Worst-case runtime $O(n^2)$
- Surely we can do better?!

Insertion sort in Practice on Worst-case Instances



n	46929	102428	364178	1014570
secs	1.03084	4.81622	61.2737	497.879

Properties of a Sorting Algorithm

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Definition (in place)

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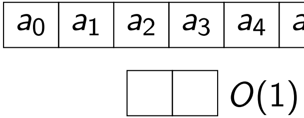
A sorting algorithm is *in place* if at any moment at most $O(1)$ array elements are stored outside the array



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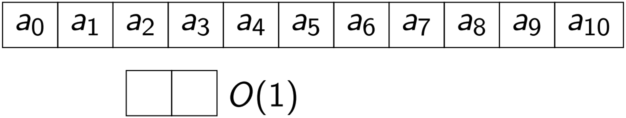

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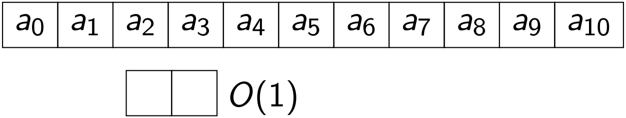

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Definition (stability)

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a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}

		$O(1)$

Example: Insertion-sort is in place

Definition (stability)
A sorting algorithm is *stable* if any pair of equal numbers in the input array appear in the same order in the sorted array

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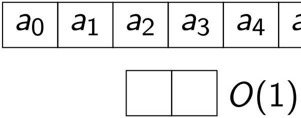

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A sorting algorithm is *stable* if any pair of equal numbers in the input array appear in the same order in the sorted array

Example: Insertion-sort is stable

Sorting Complex Data

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- In reality, data that is to be sorted is rarely entirely numerical (e.g. sort people in a database according to their last name)

Records, Keys, and Satellite Data

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family name	first name	data of birth	role
Smith	Peter	02.10.1982	lecturer
Hills	Emma	05.05.1975	reader
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...			

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Observe: Stability makes more sense when sorting complex data as opposed to numbers

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Key Idea:

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- Copy right half of A to new array C

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Merge Operation

- Copy left half of A to new array B
- Copy right half of A to new array C
- Traverse B and C simultaneously from left to right and write the smallest element at the current positions to A

Example: Merge Operation

A	1	4	9	10	3	5	7	11

Example: Merge Operation

A	1	4	9	10	3	5	7	11

B	1	4	9	10
C	3	5	7	11

Example: Merge Operation

A								
B	1	4	9	10				
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Example: Merge Operation

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Example: Merge Operation

A	1							
B	1	4	9	10				
C	3	5	7	11				

Example: Merge Operation

A	1	3						

B	1	4	9	10
C	3	5	7	11

Example: Merge Operation

A	1	3	4					
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Analysis: Merge Operation

Merge Operation

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- **Input:** An array A of integers of length n (n even) such that $A[0, \frac{n}{2} - 1]$ and $A[\frac{n}{2}, n - 1]$ are sorted
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Divide and Conquer!

Merge Sort: A Divide and Conquer Algorithm

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Require: Array A of n numbers
  if $n = 1$ **then**
    return A
  $A[0, \lfloor \frac{n}{2} \rfloor] \leftarrow \textsc{MergeSort}(A[0, \lfloor \frac{n}{2} \rfloor])$
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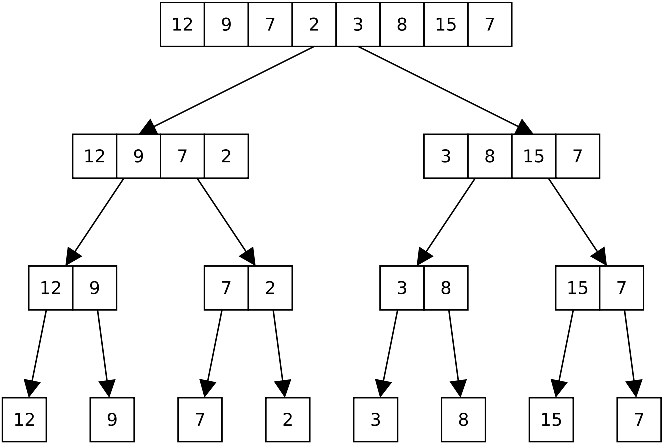
- **Combine** the solutions to the subproblems into the solution for the original problem.

Analyzing MergeSort: An Example



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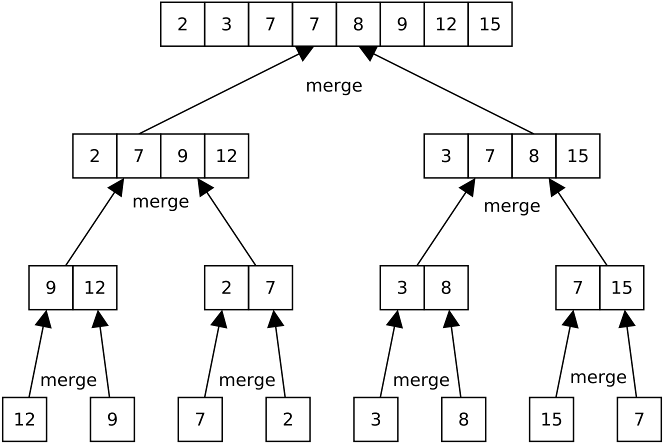

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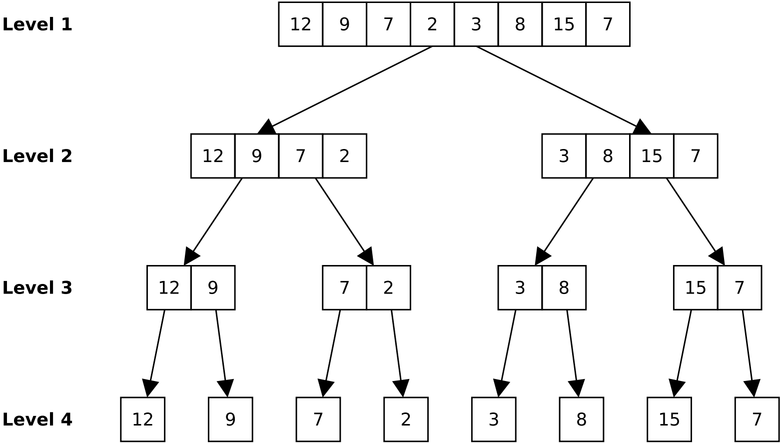
- How many nodes per level?
- Time spent per node?

Number of Levels



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$$\frac{n}{2^{l-2}} > 1$$

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$$\frac{n}{2^{l-2}} > 1 \Rightarrow n > 2^{l-2} \Rightarrow \log(n) + 2 > l$$

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Number of Levels:

- Array length in last level l is 1: $\lceil \frac{n}{2^{l-1}} \rceil = 1$

$$\frac{n}{2^{l-1}} \leq 1 \Rightarrow n \leq 2^{l-1} \Rightarrow \log(n) + 1 \leq l$$

- Array length in last but one level $l-1$ is 2: $\lceil \frac{n}{2^{l-2}} \rceil = 2$

$$\frac{n}{2^{l-2}} > 1 \Rightarrow n > 2^{l-2} \Rightarrow \log(n) + 2 > l$$

$$\log(n) + 1 \leq l < \log(n) + 2$$

Number of Levels (2)

Level i:

- 2^{i-1} nodes (at most)
- Array length in level i is $\lceil \frac{n}{2^{i-1}} \rceil$ (at most)
- Runtime of merge operation for each node in level i: $O(\frac{n}{2^{i-1}})$

Number of Levels:

- Array length in last level l is 1: $\lceil \frac{n}{2^{l-1}} \rceil = 1$

$$\frac{n}{2^{l-1}} \leq 1 \Rightarrow n \leq 2^{l-1} \Rightarrow \log(n) + 1 \leq l$$

- Array length in last but one level $l-1$ is 2: $\lceil \frac{n}{2^{l-2}} \rceil = 2$

$$\frac{n}{2^{l-2}} > 1 \Rightarrow n > 2^{l-2} \Rightarrow \log(n) + 2 > l$$

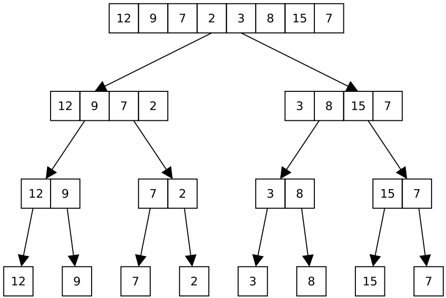
$$\log(n) + 1 \leq l < \log(n) + 2$$

Hence, $l = \lceil \log n \rceil + 1$.

Runtime of Merge Sort

Sum up Work:

- Levels:
  $l = \lceil \log n \rceil + 1$
- Nodes on level i:
  at most 2^{i-1}
- Array length in level i:
  at most $\lceil \frac{n}{2^{i-1}} \rceil$



Runtime of Merge Sort

Sum up Work:

- Levels:
  $l = \lceil \log n \rceil + 1$
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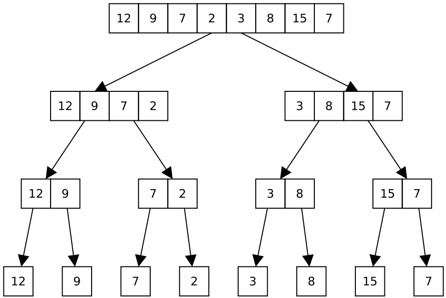

Worst-case Runtime:

Runtime of Merge Sort

Sum up Work:

- Levels:
  $l = \lceil \log n \rceil + 1$
- Nodes on level i:
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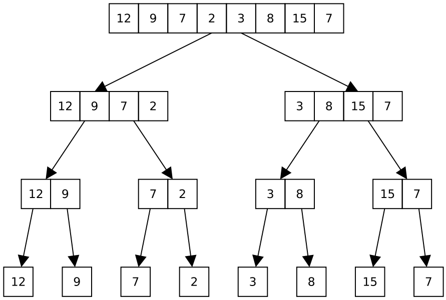

Worst-case Runtime:

$$\sum_{i=1}^{\lceil \log n \rceil + 1} 2^{i-1} O\left(\lceil \frac{n}{2^{i-1}} \rceil\right)$$

Runtime of Merge Sort

Sum up Work:

- Levels:
  $l = \lceil \log n \rceil + 1$
- Nodes on level i:
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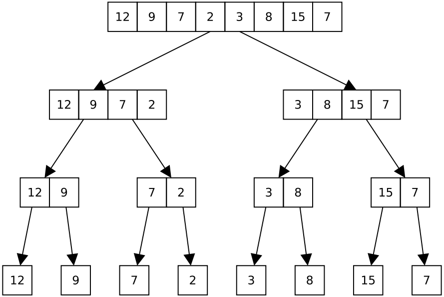

Worst-case Runtime:

$$\sum_{i=1}^{\lceil \log n \rceil + 1} 2^{i-1} O\left(\lceil \frac{n}{2^{i-1}} \rceil \right) = \sum_{i=1}^{\lceil \log n \rceil + 1} 2^{i-1} O\left(\frac{n}{2^{i-1}} \right)$$

Runtime of Merge Sort

Sum up Work:

- Levels:
  $l = \lceil \log n \rceil + 1$
- Nodes on level i:
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- Array length in level i:
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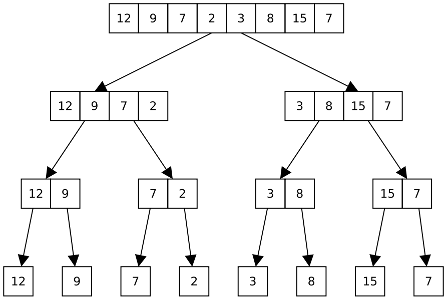

Worst-case Runtime:

$$\sum_{i=1}^{\lceil \log n \rceil + 1} 2^{i-1} O\left(\lceil \frac{n}{2^{i-1}} \rceil\right) = \sum_{i=1}^{\lceil \log n \rceil + 1} 2^{i-1} O\left(\frac{n}{2^{i-1}}\right)$$

$$= \sum_{i=1}^{\lceil \log n \rceil + 1} O(n)$$

Runtime of Merge Sort

Sum up Work:

- Levels:
  $l = \lceil \log n \rceil + 1$
- Nodes on level i:
  at most 2^{i-1}
- Array length in level i:
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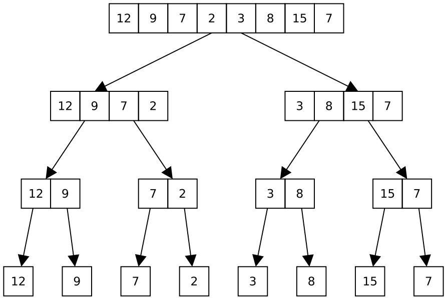

Worst-case Runtime:

$$\sum_{i=1}^{\lceil \log n \rceil + 1} 2^{i-1} O\left(\lceil \frac{n}{2^{i-1}} \rceil\right) = \sum_{i=1}^{\lceil \log n \rceil + 1} 2^{i-1} O\left(\frac{n}{2^{i-1}}\right)$$

$$= \sum_{i=1}^{\lceil \log n \rceil + 1} O(n) = (\lceil \log n \rceil + 1) O(n)$$

Runtime of Merge Sort

Sum up Work:

- Levels:
  $l = \lceil \log n \rceil + 1$
- Nodes on level i:
  at most 2^{i-1}
- Array length in level i:
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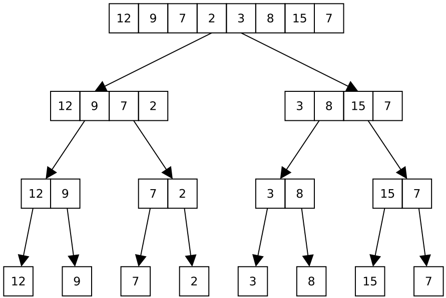
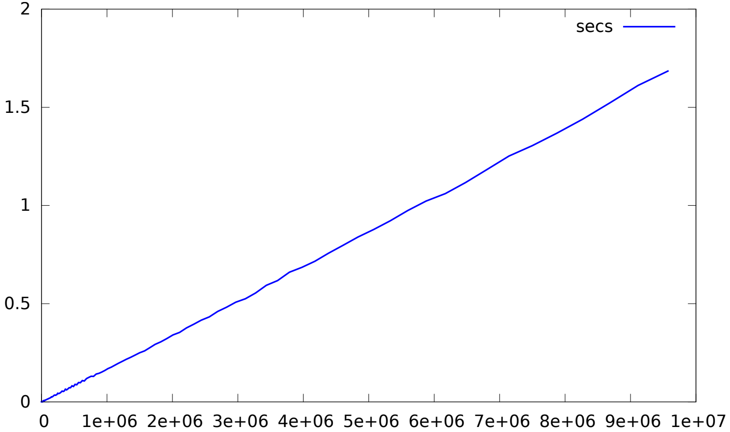


Worst-case Runtime:

$$\sum_{i=1}^{\lceil \log n \rceil + 1} 2^{i-1} O\left(\lceil \frac{n}{2^{i-1}} \rceil\right) = \sum_{i=1}^{\lceil \log n \rceil + 1} 2^{i-1} O\left(\frac{n}{2^{i-1}}\right)$$

$$= \sum_{i=1}^{\lceil \log n \rceil + 1} O(n) = (\lceil \log n \rceil + 1)\, O(n) = O(n \log n) \ .$$

Merge sort in Practice on Worst-case Instances



n	46929	102428	364178	1014570
secs	1.03084	4.81622	61.2737	497.879 (Insertion-sort)
secs	0.007157	0.015802	0.0645791	0.169165 (Merge-sort)

Stability and In Place Property?

Stability and In Place Property?

Stability and In Place Property?

Stability and In Place Property?

- Merge sort is stable

Stability and In Place Property?

Stability and In Place Property?

- Merge sort is stable
- Merge sort does not sort in place

Generalizing the Analysis

Divide and Conquer Algorithm:

Generalizing the Analysis

Divide and Conquer Algorithm:

Let **A** be a divide and conquer algorithm with the following properties:

Generalizing the Analysis

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1. **A** performs two recursive calls on input sizes at most $n/2$

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Generalizing the Analysis

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2. The conquer operation in **A** takes $O(n)$ time

Then:

Generalizing the Analysis

Divide and Conquer Algorithm:

Let **A** be a divide and conquer algorithm with the following properties:

1. **A** performs two recursive calls on input sizes at most $n/2$
2. The conquer operation in **A** takes $O(n)$ time

Then:

A has a runtime of $O(n \log n)$.