Heap Sort
COMS10017 - Algorithms 1

Dr Christian Konrad

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- Building block of many efficient algorithms
- For example, an array is a data structure

Priority Queues

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Data structure that allows the following operations:

- Build(.): Create data structure given a set of data items
- Extract-Max(.): Remove the maximum element from the data structure and return it
- *others...*

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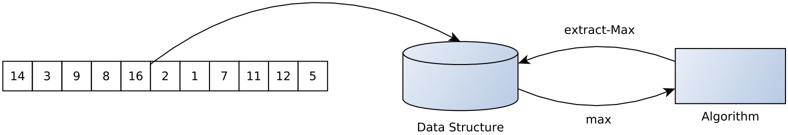
Sorting using a Priority Queue

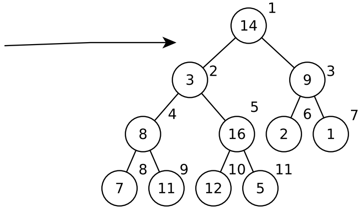


From Array to Tree

Interpretation of an Array as a Complete Binary Tree

Interpretation of an Array as a Complete Binary Tree

1	2	3	4	5	6	7	8	9	10	11
14	3	9	8	16	2	1	7	11	12	5



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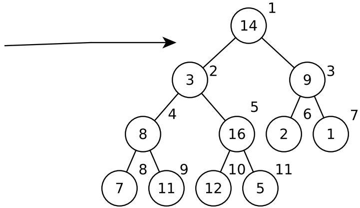

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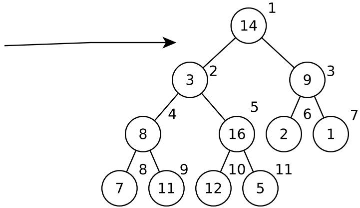

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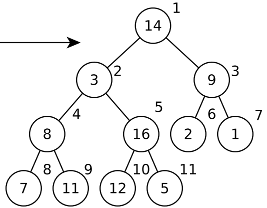
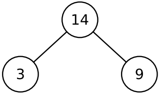


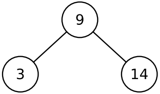
Easy Navigation:

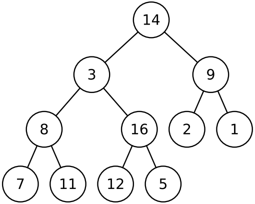
- Parent of i: $\lfloor i/2 \rfloor$
- Left/Right Child of i: $2i$ and $2i+1$

The Heap Property

Key of nodes larger than keys of their children







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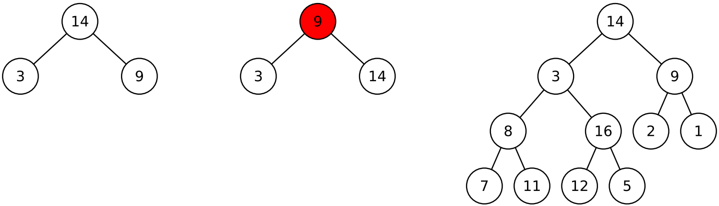

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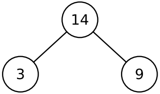

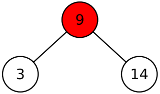

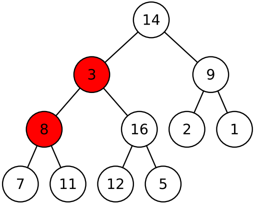

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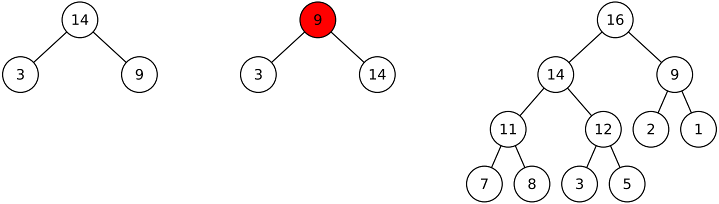

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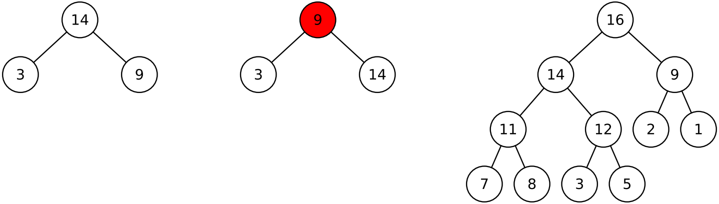

Heap Property → Maximum at root
Important for Extract-Max(.)

The Heapify Operation

Constructing a Heap: Build(.)
Given a binary tree, transform it into one that fulfills the Heap Property

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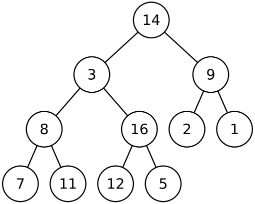

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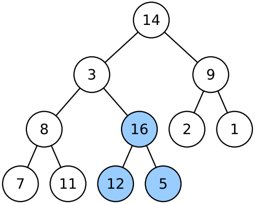

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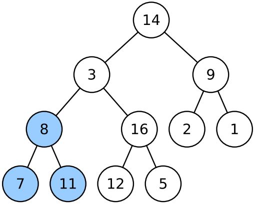

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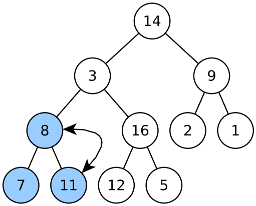

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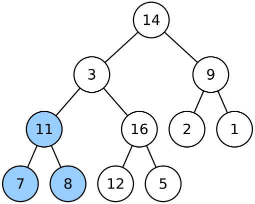

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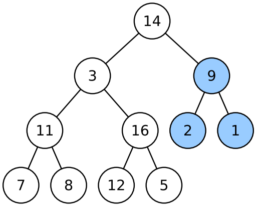

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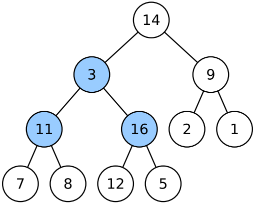

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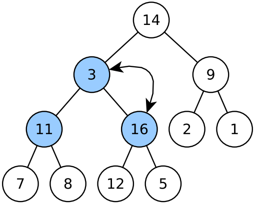

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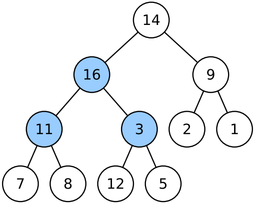

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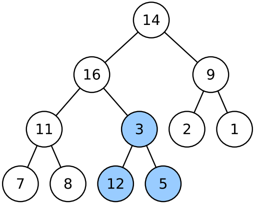

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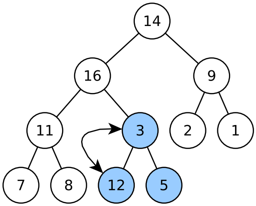

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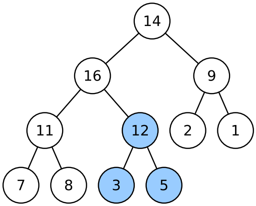

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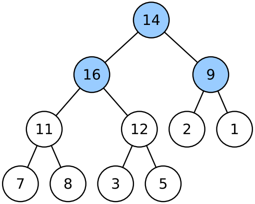

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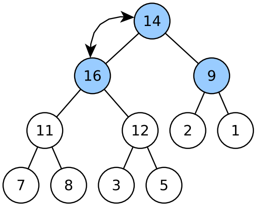

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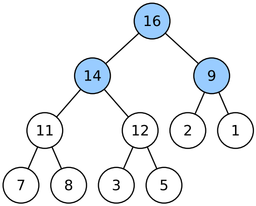

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Constructing a Heap: Build(.) Runtime $O(n \log n)$

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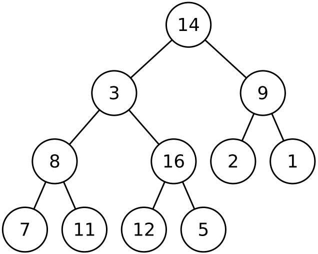

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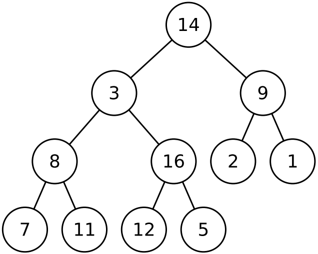

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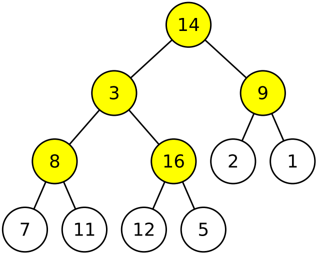

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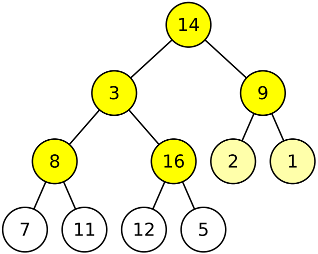

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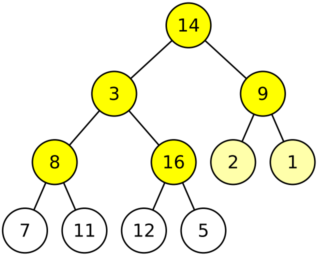
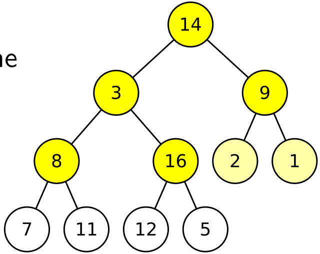
- These nodes are contained in a perfect binary tree
- This tree has i levels



Analysis

We sum over all relevant levels, count the number of nodes per level, and multiply with the depth of their subtrees:



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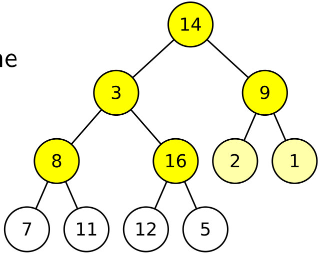

$$\text{Runtime} = \sum_{j=1}^{i} \#\ \text{nodes at level } (i-j+1) \cdot \text{depth of subtree} \cdot O(1)$$

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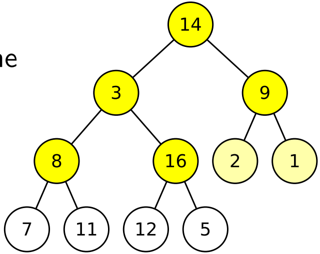
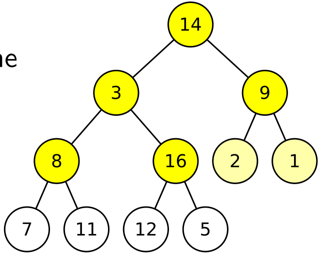


$$\text{Runtime} = \sum_{j=1}^{i} \# \text{ nodes at level } (i-j+1) \cdot \text{depth of subtree} \cdot O(1)$$
$$= O(1) \sum_{j=1}^{i} 2^{i-j} \cdot j$$

Improved Analysis of Heap Construction

Analysis
We sum over all relevant levels, count the number of nodes per level, and multiply with the depth of their subtrees:



$$\text{Runtime} = \sum_{j=1}^{i} \# \text{ nodes at level } (i-j+1) \cdot \text{depth of subtree} \cdot O(1)$$

$$= O(1) \sum_{j=1}^{i} 2^{i-j} \cdot j = O(1) \cdot 2^i \cdot \sum_{j=1}^{i} \frac{j}{2^j}$$

Improved Analysis of Heap Construction

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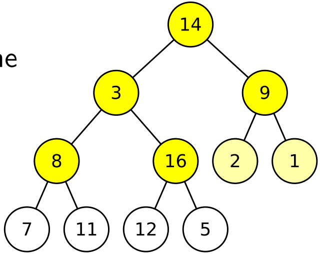

$$\begin{aligned}
\text{Runtime} &= \sum_{j=1}^{i} \#\text{ nodes at level } (i-j+1) \cdot \text{depth of subtree} \cdot O(1) \\
&= O(1) \sum_{j=1}^{i} 2^{i-j} \cdot j = O(1) \cdot 2^i \cdot \sum_{j=1}^{i} \frac{j}{2^j} \\
&= O(2^i)
\end{aligned}$$

Improved Analysis of Heap Construction

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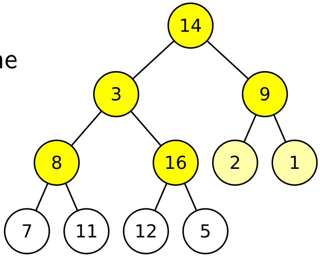

$$\begin{aligned}\text{Runtime} &= \sum_{j=1}^{i} \#\text{ nodes at level } (i-j+1) \cdot \text{depth of subtree} \cdot O(1) \\ &= O(1)\sum_{j=1}^{i} 2^{i-j} \cdot j = O(1) \cdot 2^i \cdot \sum_{j=1}^{i} \frac{j}{2^j} \\ &= O(2^i) = O(n')\end{aligned}$$

Improved Analysis of Heap Construction

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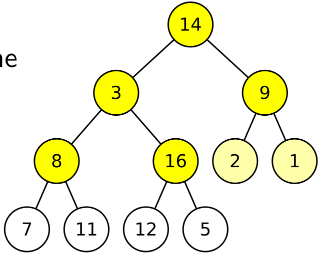

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Improved Analysis of Heap Construction

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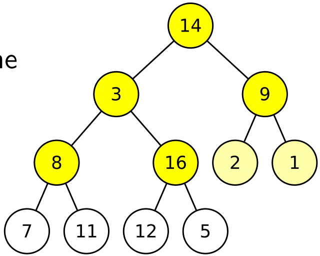

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&= O(2^i) = O(n') = O(n) \ ,
\end{aligned}$$

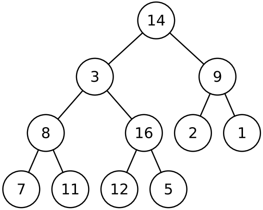
using $\sum_{j=1}^{i} \frac{j}{2^j} = O(1)$ (see trick from linear/binary search lecture).

The Complete Algorithm

Putting Everything Together

14	3	9	8	16	2	1	7	11	12	5

1. Build()
2. Repeat *n* times:
   1. Swap root with last element
   2. Decrease size of heap by 1
   3. Heapify(root)



The Complete Algorithm

Putting Everything Together

16	14	9	11	12	2	1	7	8	3	5

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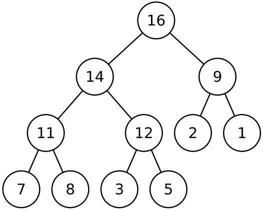

Putting Everything Together

5	14	9	11	12	2	1	7	8	3	16

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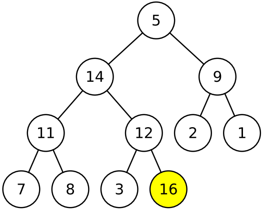

The Complete Algorithm

Putting Everything Together

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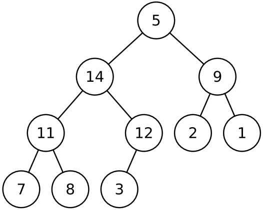

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The Complete Algorithm

Putting Everything Together

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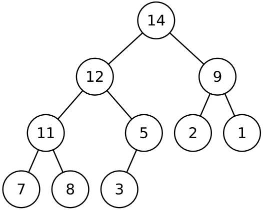
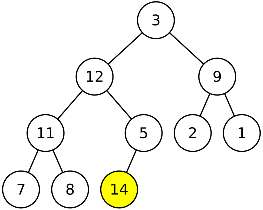


Putting Everything Together

3	12	9	11	5	2	1	7	8	14	16



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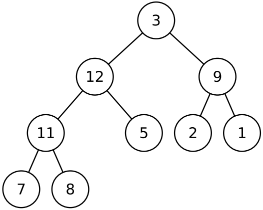
The Complete Algorithm

Putting Everything Together

3	12	9	11	5	2	1	7	8	14	16



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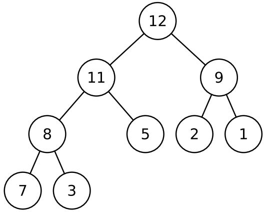
The Complete Algorithm

Putting Everything Together

12	11	9	8	5	2	1	7	3	14	16



1. Build()
2. Repeat n times:
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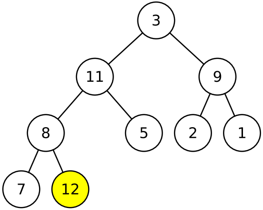
The Complete Algorithm

Putting Everything Together

3	11	9	8	5	2	1	7	12	14	16



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The Complete Algorithm

Putting Everything Together

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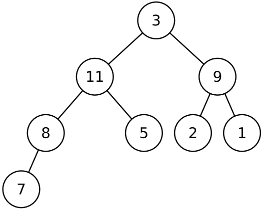

The Complete Algorithm

Putting Everything Together

3	11	9	8	5	2	1	7	12	14	16

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   1. Swap root with last element
   2. Decrease size of heap by 1
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...

The Complete Algorithm

Putting Everything Together

1	2	3	5	7	8	9	11	12	14	16

1. Build()
2. Repeat n times:
    1. Swap root with last element
    2. Decrease size of heap by 1
    3. Heapify(root)

The Complete Algorithm

Putting Everything Together

1	2	3	5	7	8	9	11	12	14	16

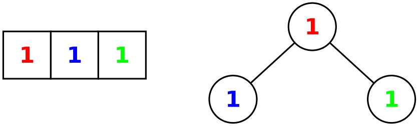
1. Build() $O(n)$
2. Repeat n times:
   1. Swap root with last element $O(1)$
   2. Decrease size of heap by 1 $O(1)$
   3. Heapify(root) $O(\log n)$

Runtime: $O(n \log n)$

Heapsort is Not Stable

Example:

1. Build()
2. Repeat *n* times:
    1. Swap root with last element
    2. Decrease size of heap by 1
    3. Heapify(root)



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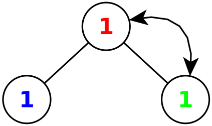
   3. Heapify(root)

1	1	1



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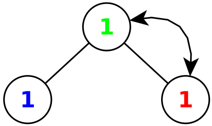

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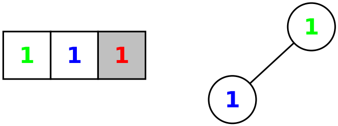

1 is moved from left to the right past 1 and 1

Heap-sort not stable