Quicksort

COMS10017 - Algorithms 1

Dr Christian Konrad

Quicksort

Sorting Algorithms seen so far:

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- *In place version of Mergesort:*

$A[0, \lfloor \frac{n}{2} \rfloor] \leftarrow \textsc{MergeSort}(A[0, \lfloor \frac{n}{2} \rfloor])$
$A[\lfloor \frac{n}{2} \rfloor + 1, n-1] \leftarrow \textsc{MergeSort}(A[\lfloor \frac{n}{2} \rfloor, n-1])$
$A \leftarrow \textsc{Merge}(A)$
return A

recursive calls in mergesort

Merge Sort versus Quick Sort

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- *Mergesort:* First solve subproblems recursively, then merge their solutions
- *Quicksort:* First partition problem into two subproblems in a clever way so that no extra work is needed when combining the solutions to the subproblems, then solve subproblems recursively

Divide and Conquer Algorithm:

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- **Divide:** Chose a good *pivot* $A[q]$. Rearrange A such that every element $\leq A[q]$ is left of $A[q]$ in the resulting ordering and every element $> A[q]$ is right of $A[q]$ in the resulting ordering. Let p be the new position of $A[q]$.

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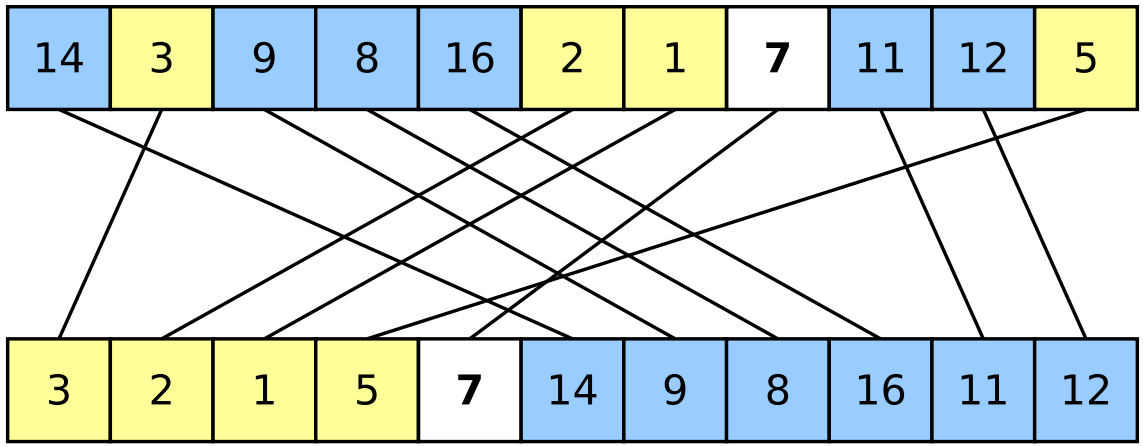

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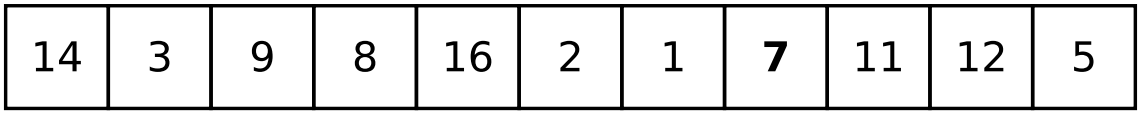

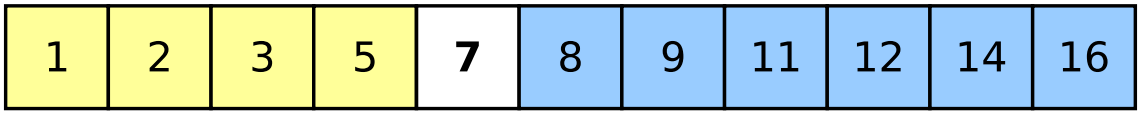

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- **Combine:** No work needed

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2. What is a good pivot? Ideally we would like to obtain subproblems of equal/similar sizes

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PARTITION

Pivot: Algorithm assumes pivot is $A[n-1]$ (if different pivot $A[q]$ is used: swap $A[q]$ with $A[n-1]$).

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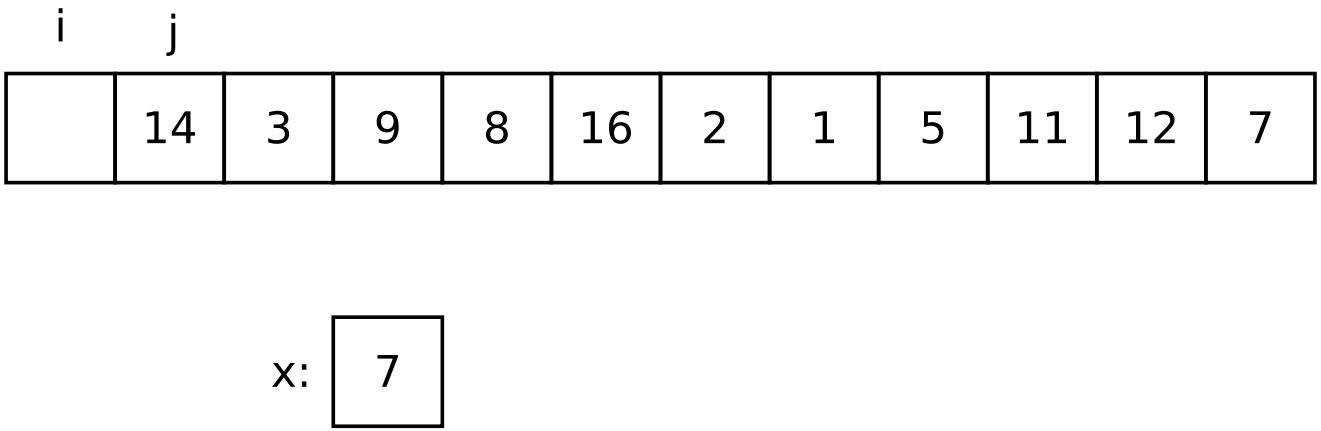

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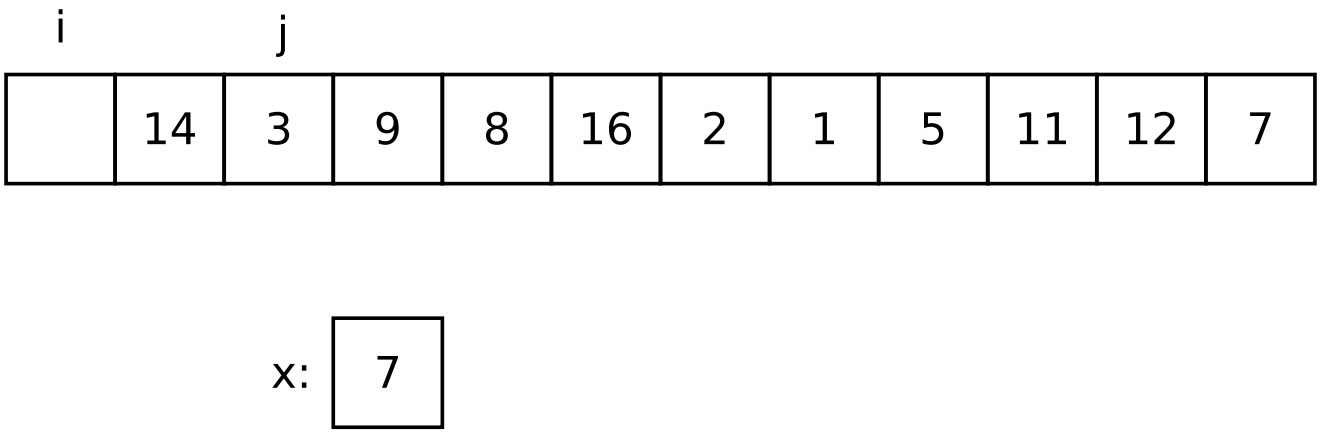

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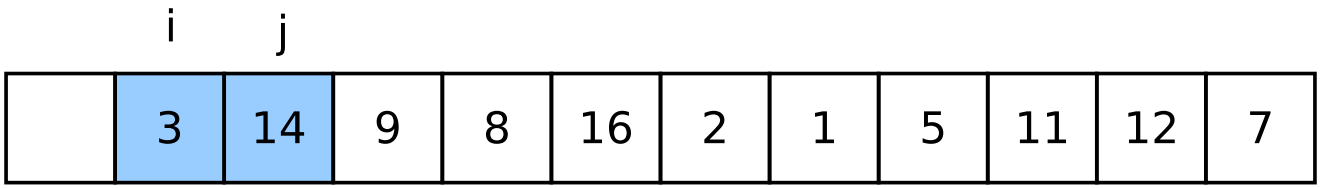

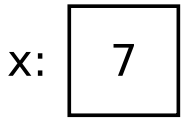

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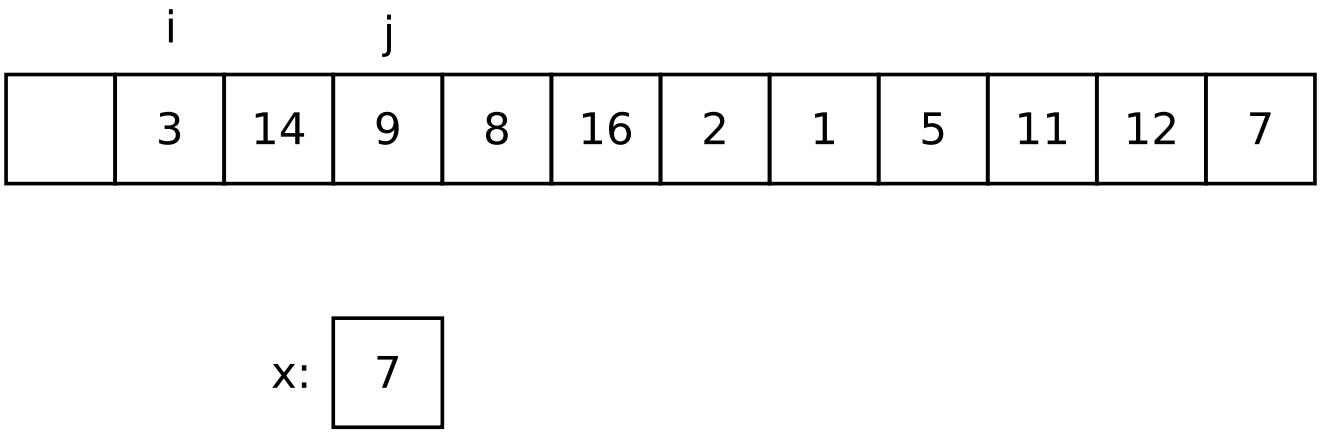

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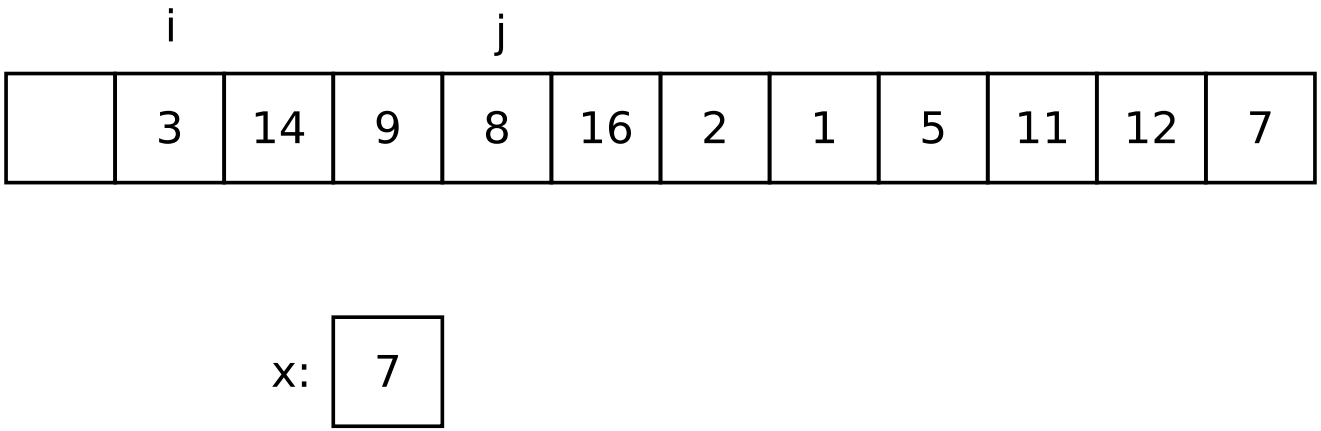

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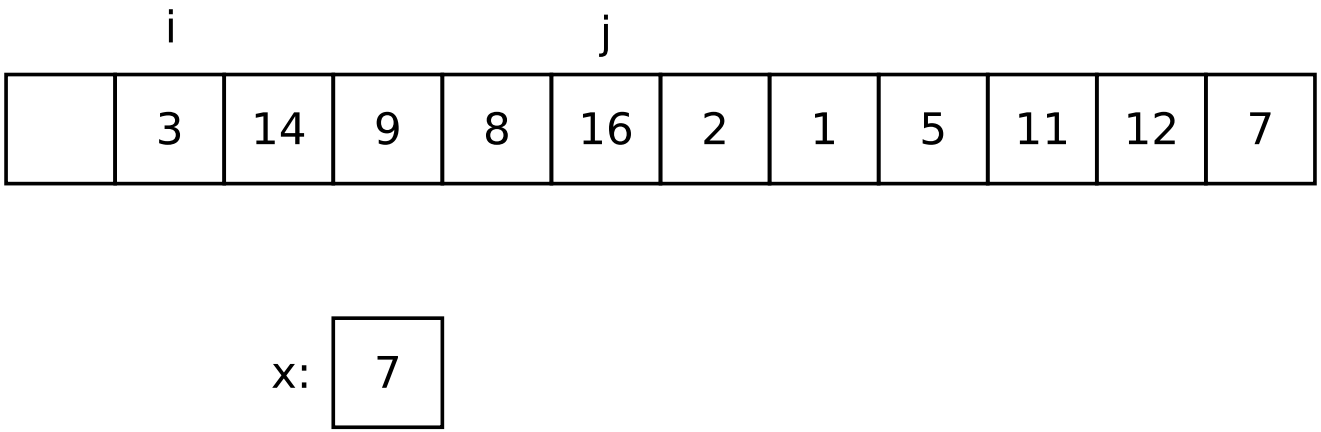

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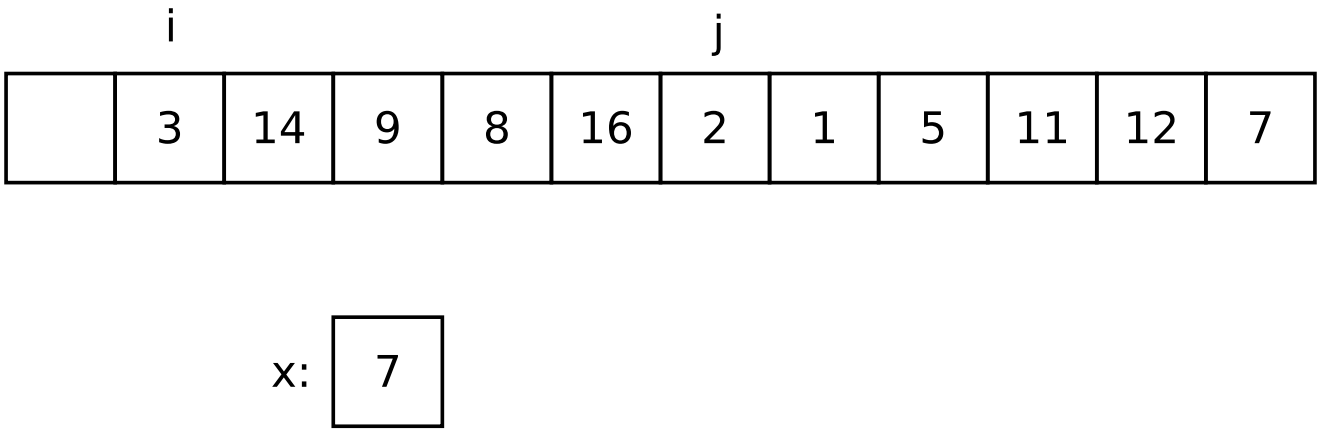

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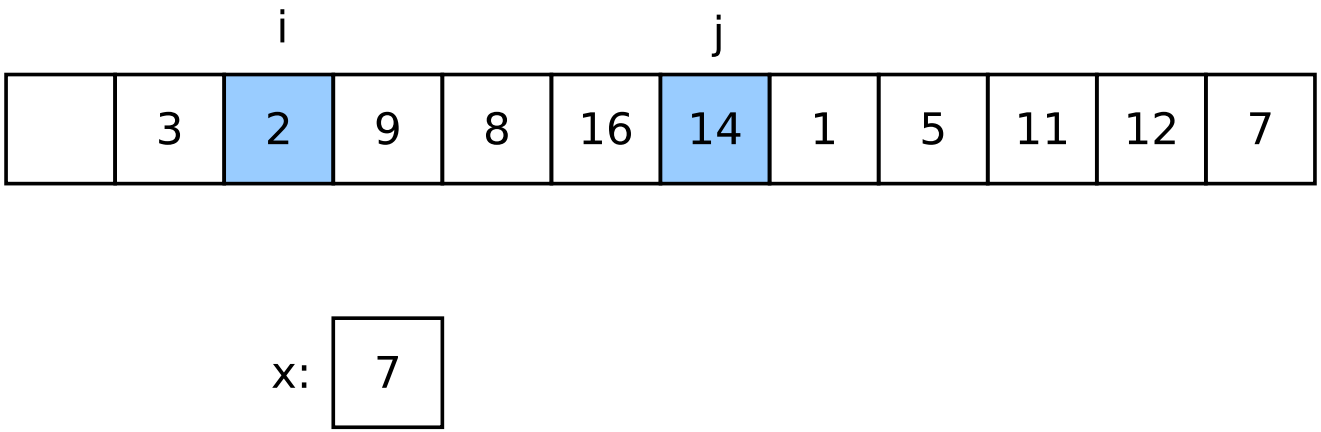

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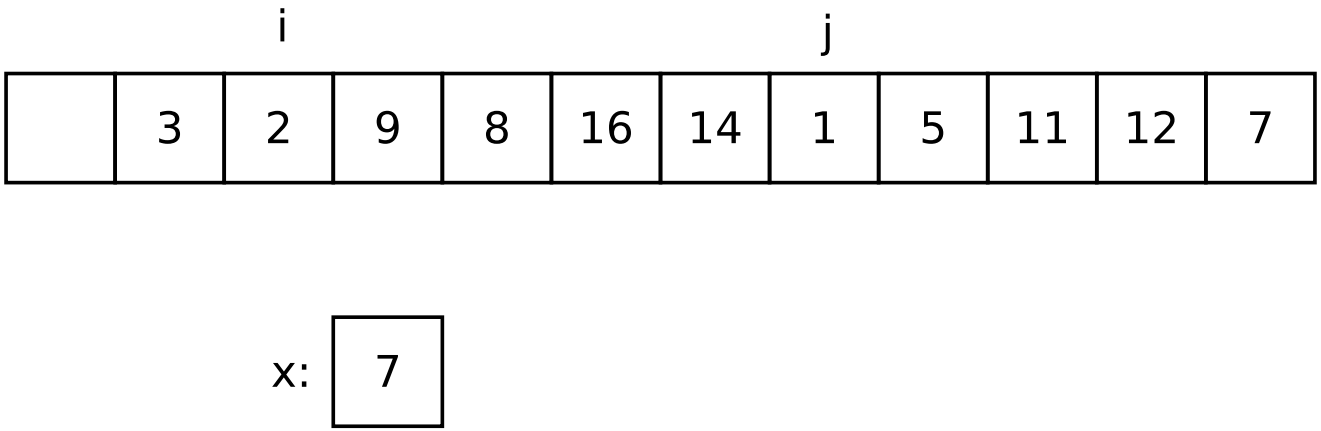

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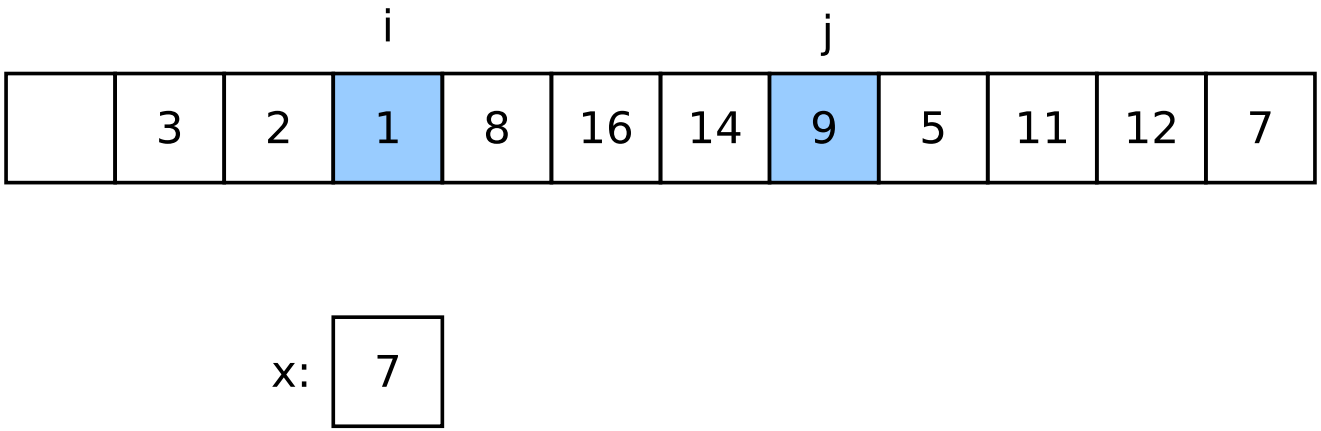

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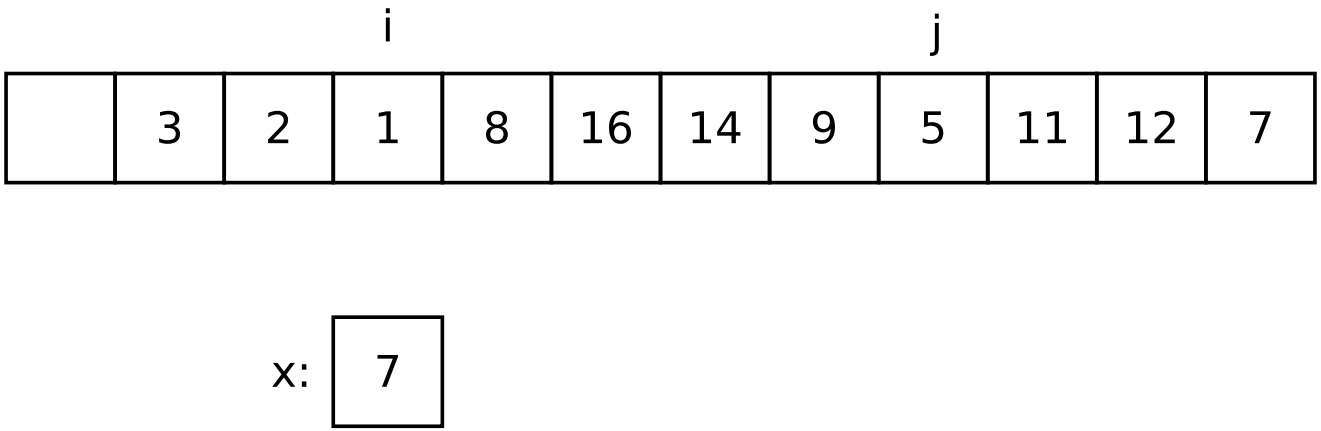

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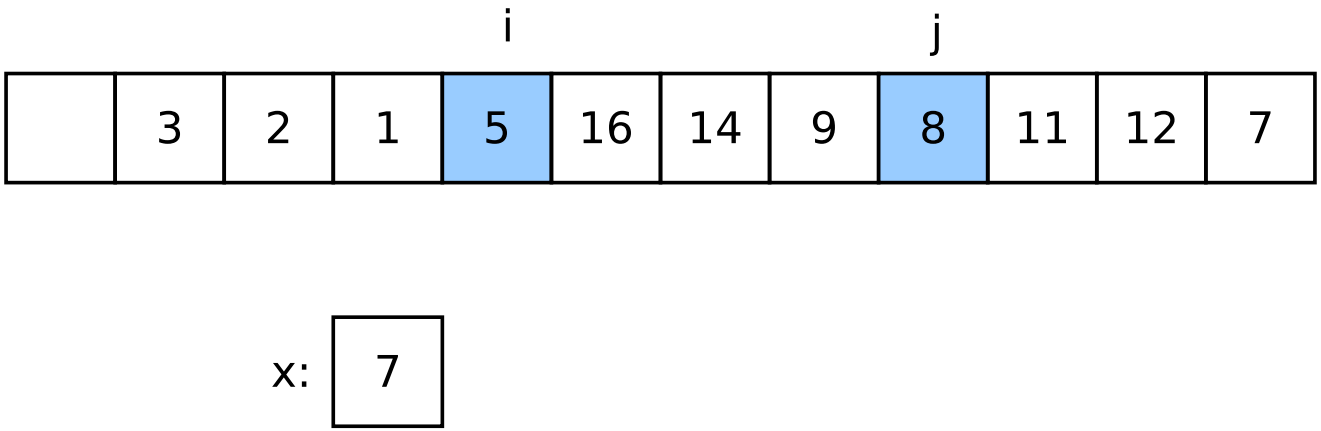

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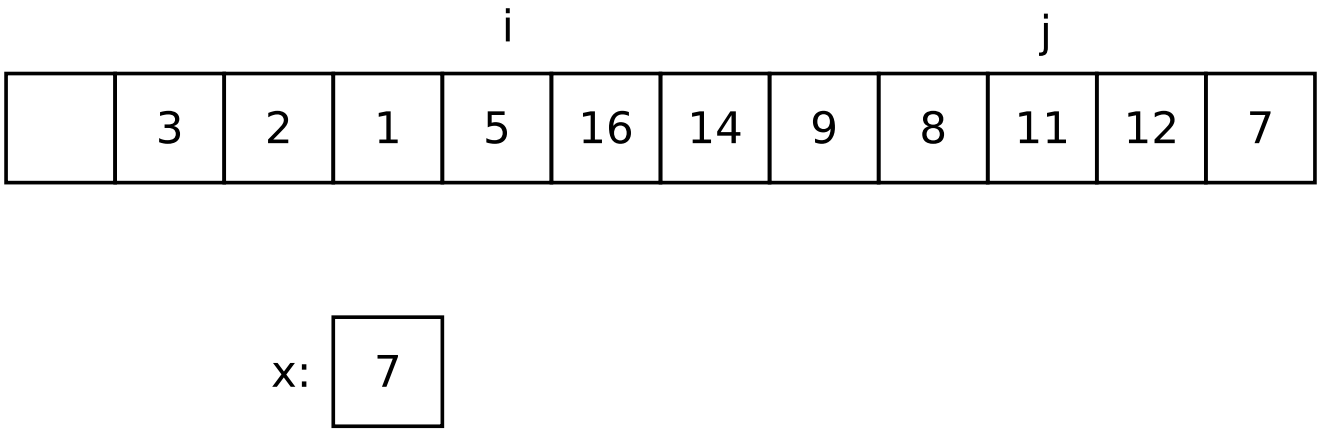

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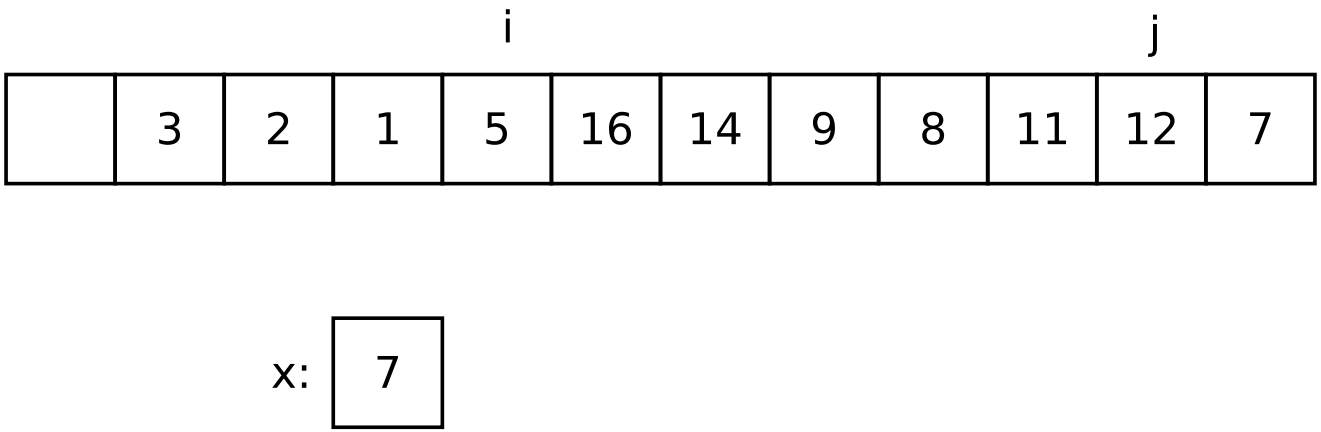

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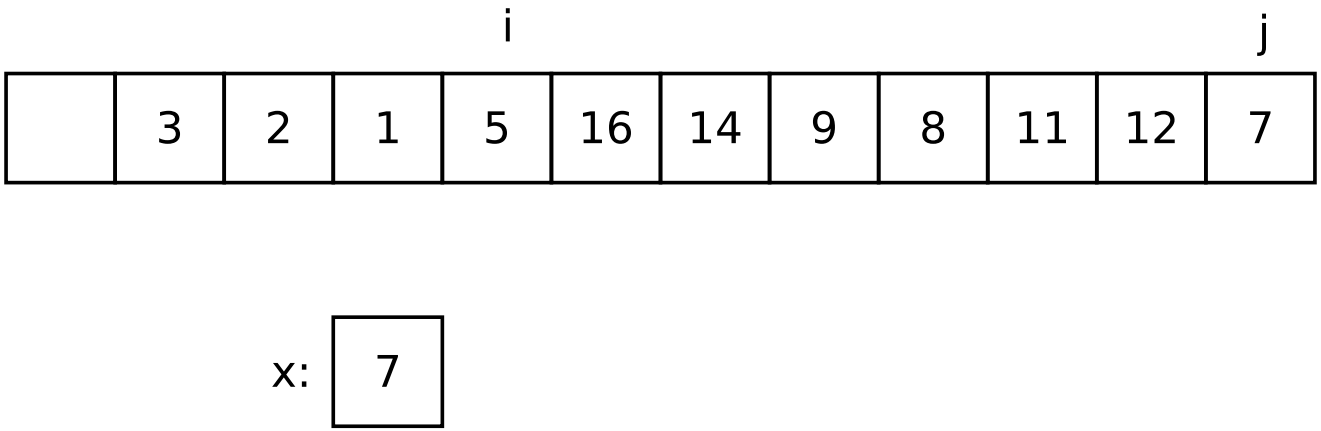

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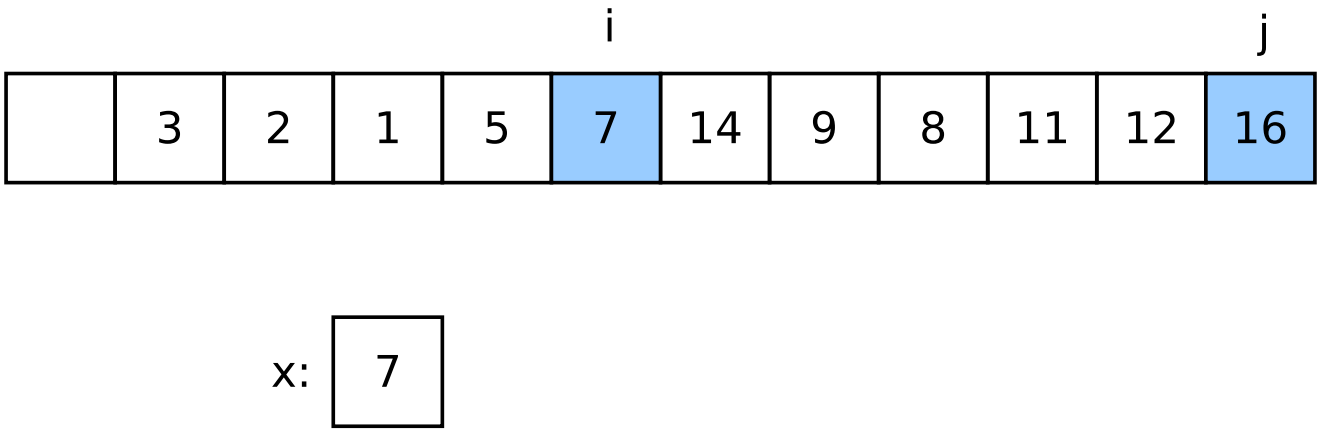

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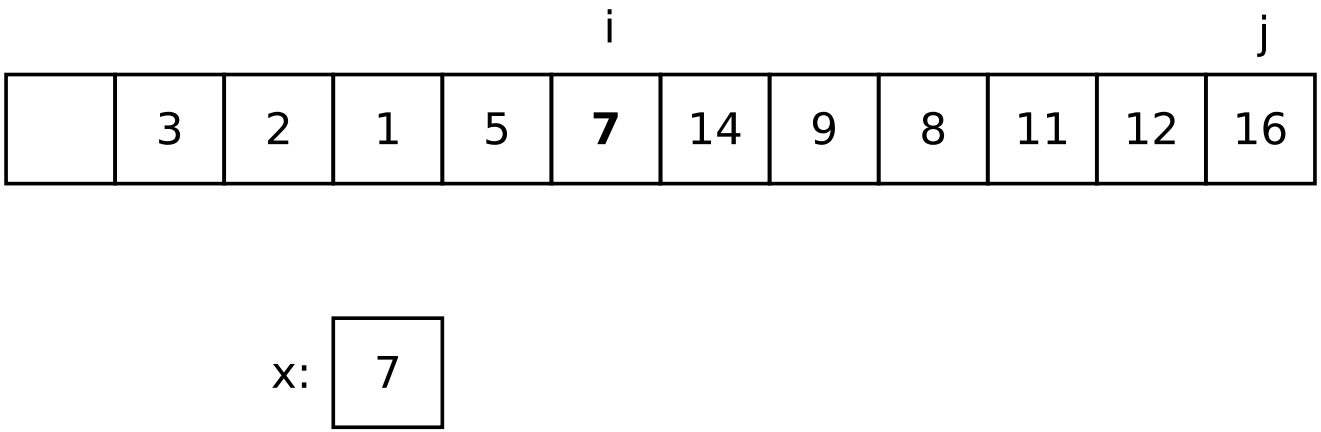

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2. Elements right of i (excluding i) and left of j (excluding j) are larger than x:

$$\text{For } i+1 \leq k \leq j-1 : A[k] > x$$

Proof of Loop Invariant

1. Left of i (including i): smaller equal to x
2. Right of i and left of j (excl. j): larger than x

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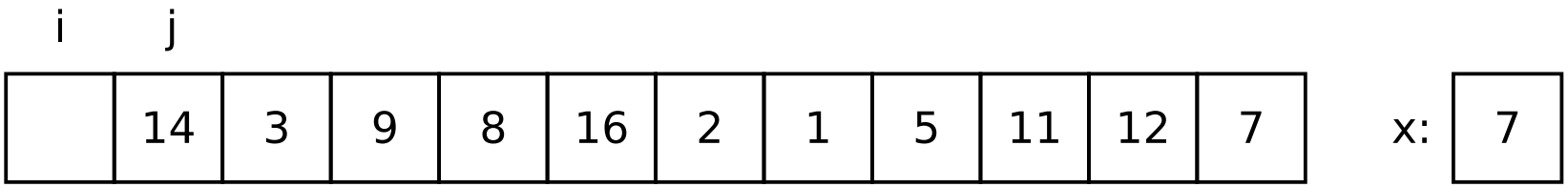

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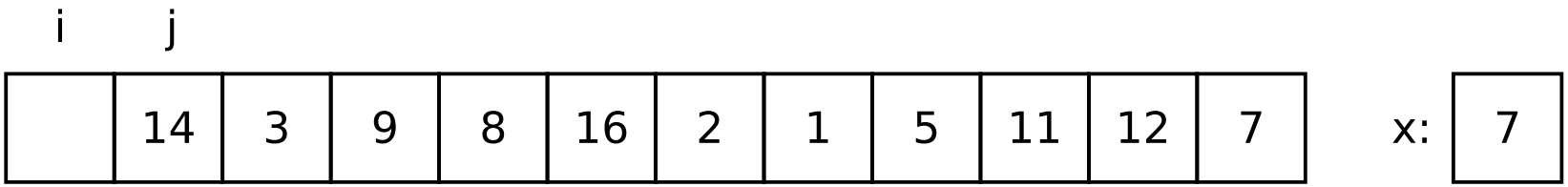

Proof of Loop Invariant (2)

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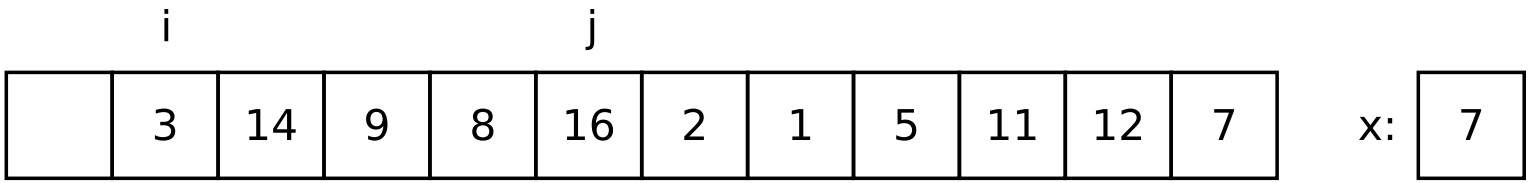

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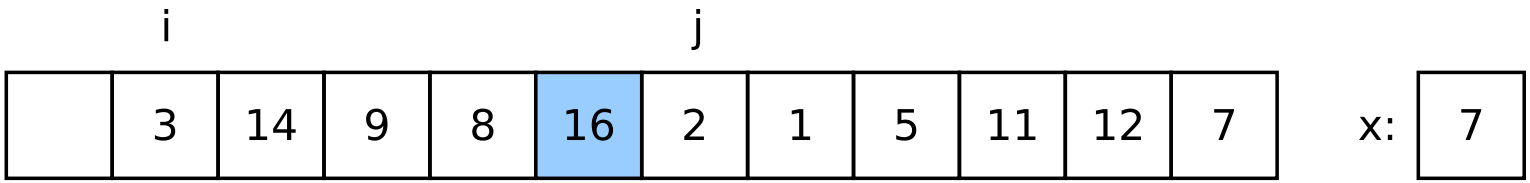

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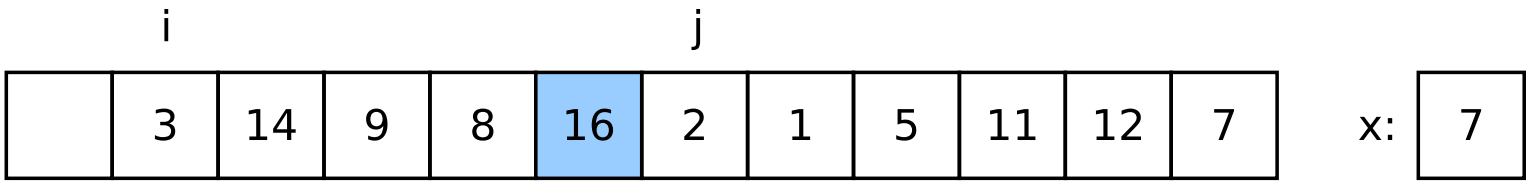

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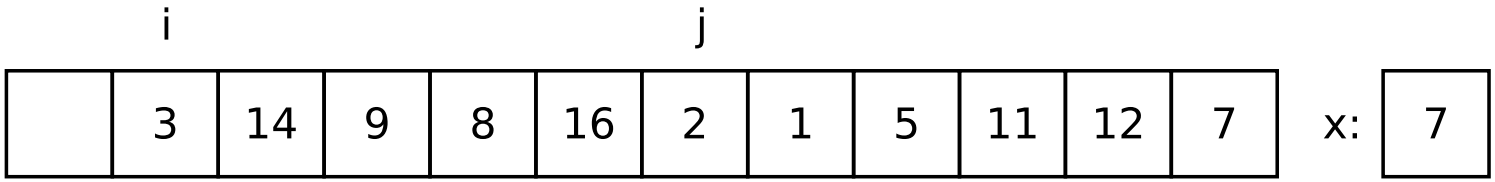
2. $A[j] \leq x$: Then i is incremented, $A[i]$ and $A[j]$ are exchanged, and j is incremented



Proof of Loop Invariant (2)

1. Left of i (including i): smaller equal to x
2. Right of i and left of j (excl. j): larger than x

```
x ← A[n − 1]
i ← −1
for j ← 0 . . . n − 1 do
    if A[j] ≤ x then
        i ← i + 1
        exchange A[i] with A[j]
```

Maintenance: Two cases:

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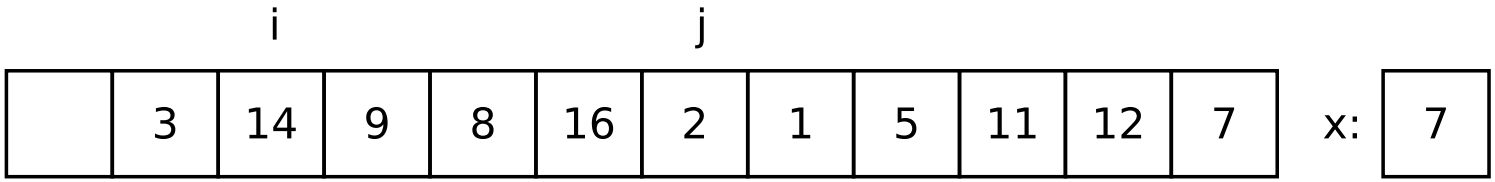

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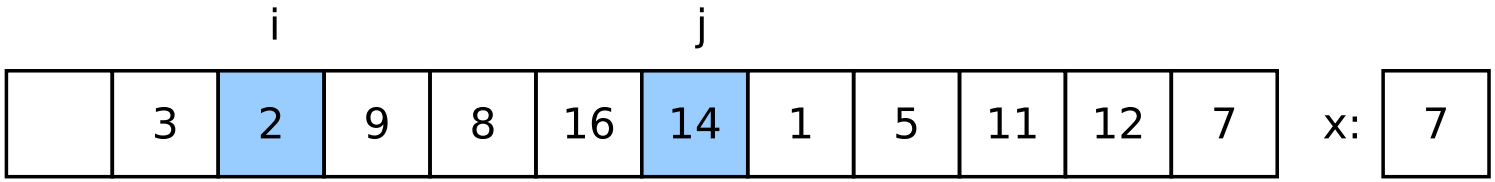

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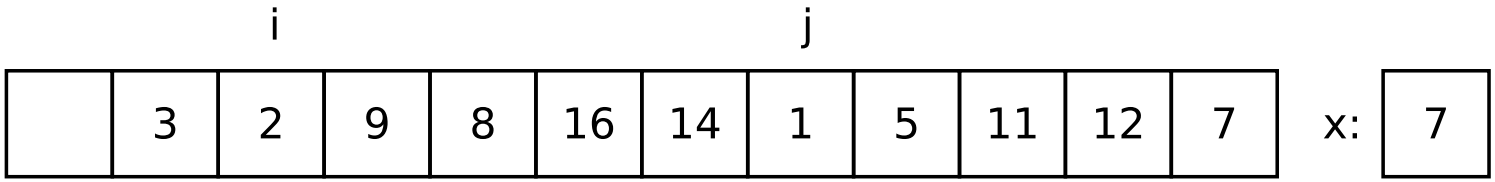

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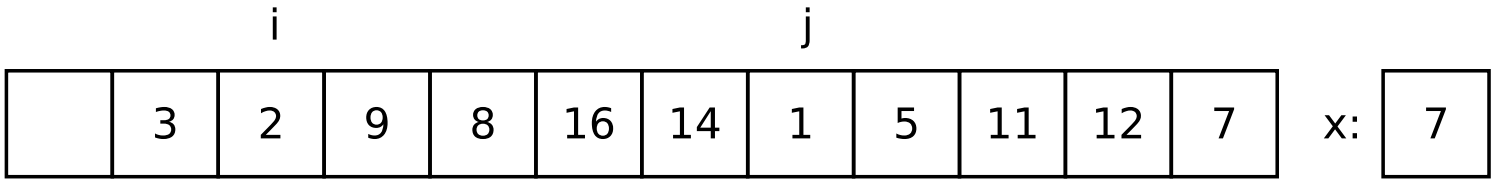

Proof of Loop Invariant (3)

1. Left of i (including i): smaller equal to x
2. Right of i and left of j (excl. j): larger than x

```
x ← A[n − 1]
i ← −1
for j ← 0 . . . n − 1 do
    if A[j] ≤ x then
        i ← i + 1
        exchange A[i] with A[j]
```

Termination: (useful property showing that algo. is correct)

- $A[i]$ contains pivot element x that was located initially at position $n-1$
- All elements left of $A[i]$ are smaller equal to x
- All elements right of $A[i]$ are larger than x

Proof of Loop Invariant (3)

1. Left of i (including i): smaller equal to x
2. Right of i and left of j (excl. j): larger than x

$x \leftarrow A[n-1]$
$i \leftarrow -1$
for $j \leftarrow 0 \ldots n-1$ **do**
  if $A[j] \leq x$ **then**
    $i \leftarrow i+1$
    exchange $A[i]$ with $A[j]$

Termination: (useful property showing that algo. is correct) ✓

- $A[i]$ contains pivot element x that was located initially at position $n-1$
- All elements left of $A[i]$ are smaller equal to x
- All elements right of $A[i]$ are larger than x

Quicksort

Require: array A of length n
  if $n \leq 1$ **then**
    return A
  else
    $i \leftarrow \text{Partition}(A)$
    $\textsc{Quicksort}(A[0, i-1])$
    $\textsc{Quicksort}(A[i+1, n-1])$

Algorithm $\textsc{Quicksort}$

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What is the runtime of Quicksort?