Lower Bound for Sorting

COMS10017 - (Object-Oriented Programming and) Algorithms

Dr Christian Konrad

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- Count number of 0s n_0
- Write n_0 0s followed by $n - n_0$ 1s
- Both operations take time $O(n)$

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Lower Bound for Comparison-based Sorting

- We will prove that every comparison-based sorting algorithm requires $\Omega(n \log n)$ comparisons
- This implies that $O(n \log n)$ is an optimal runtime for comparison-based sorting

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- A *bijective* function $\pi : [n] \rightarrow [n]$ is called a permutation

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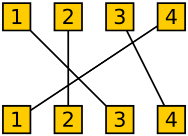

$$\begin{aligned} \pi(1) &= 3 \\ \pi(2) &= 2 \\ \pi(3) &= 4 \\ \pi(4) &= 1 \end{aligned}$$

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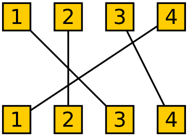

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- A reordering of $[n]$

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Lemma

$|\Pi| = n! = n \cdot (n-1) \dots 3 \cdot 2 \cdot 1$

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Rephrasing our Task:

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Rephrasing our Task: Find permutation $\pi \in \Pi$ such that:

$$A[\pi^{-1}(1)] < A[\pi^{-1}(2)] < \cdots < A[\pi^{-1}(n-1)] < A[\pi^{-1}(n)]$$

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At least 3 *queries are needed to sort* 3 *elements.*

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Lemma

At least 3 queries are needed to sort 3 elements.

Proof. Let the three elements be a, b, c.

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Proof. Let the three elements be a, b, c. Suppose that the first query is $a < b$ and suppose that the answer is yes.

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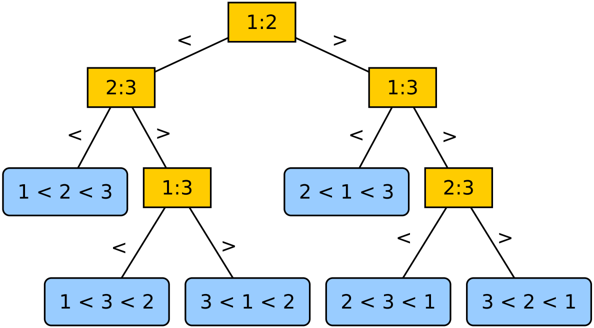
Next we either ask $a < c$ or $b < c$. Suppose that we ask $a < c$. Then, if the answer is yes then we are left with cases 1 and 2 and we need an additional query. Suppose that we ask $b < c$. Then, if the answer is no then we are left with cases 2 and 3 and we need an additional query. $\square$

Decision-tree Model (2)

Every Guessing Strategy (and Sorting Algorithm) is a Decision-tree



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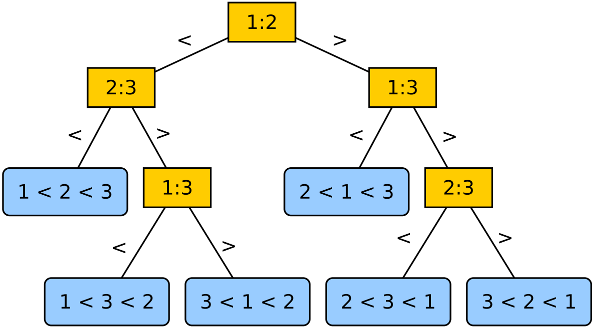

Observe:

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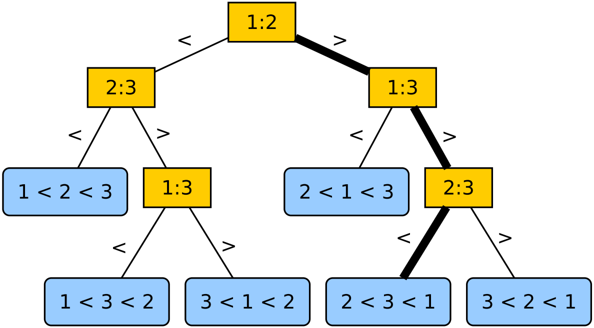

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- An execution is a root-to-leaf path

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Any comparison-based sorting algorithm requires $\Omega(n \log n)$ *comparisons.*

Proof Observe that decision-tree is a binary tree. Every potential permutation is a leaf. There are $n!$ leaves. A binary tree of height h has no more than 2^h leaves.

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$\square$

Stirling's approximation: $n! \geq \left(\frac{n}{e}\right)^n$