Recurrences II

COMS10017 - Algorithms 1

Dr Christian Konrad

Recursion Tree Method

Recursion Tree:

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- Each node represents cost of single subproblem

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$T(64)$

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$$T(64) = 2T(16) + 32$$

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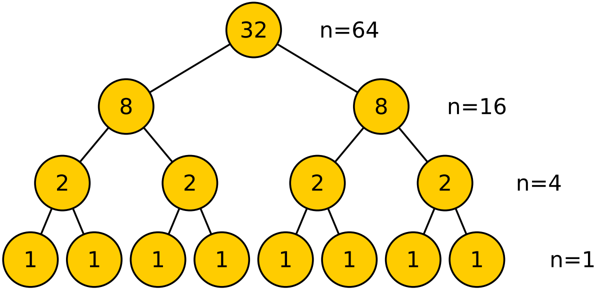

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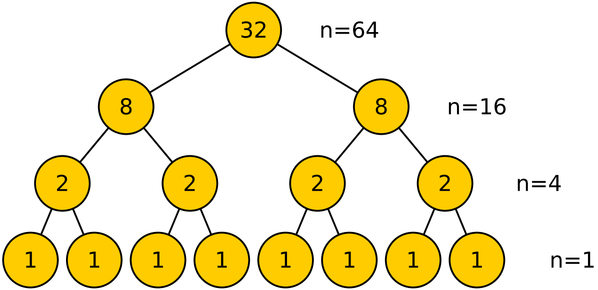

Sum of values assigned to nodes equals $T(64)$

Obtaining a Good Guess for Solution

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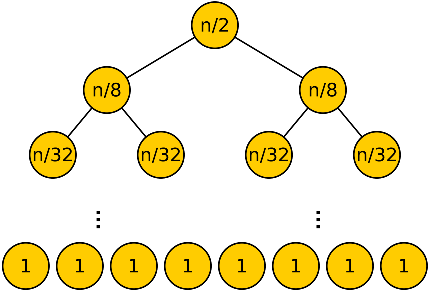

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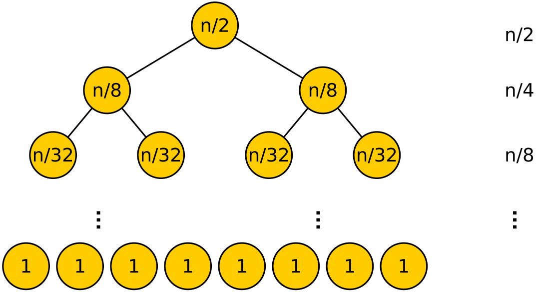

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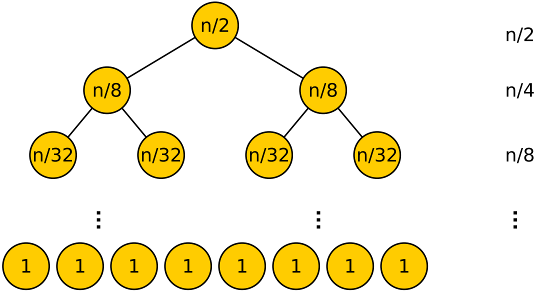

Sum of Nodes in Level i**:** $\frac{n}{2^i}$ (except the last level)

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Use substitution method to prove that guess is correct!

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Verify the Base Case: $T(1) = 1 \leq c \cdot 1 = c$ for every $c \geq 1$.

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Summary:

- We proved $T(n) \le n$, for every $n \ge 1$
- Hence $T(n) \in O(n)$

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Substitution Method

- Guess correct solution
- Verify guess using mathematical induction
- Guessing can be difficult and requires experience