Θ and Ω Notation

COMS10018 - Algorithms

Dr Christian Konrad

Limitations/Strengths of Big-O

O-notation: Upper Bound

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How to Avoid Ambiguities

- Θ-notation: Growth is precisely determined (up to constants)
- Ω-notation: Gives us a lower bound (up to constants)

Θ-notation

"Theta"-notation:
Growth is precisely determined up to constants

Definition: Θ-notation ("Theta")

Let $g(n)$ be a function. Then $\Theta(g(n))$ is the set of functions:

$$\Theta(g(n)) = \{f(n) : \text{There exist positive constants } c_1, c_2 \text{ and } n_0 \text{ s.t. } 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \text{ for all } n \geq n_0\}$$

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$f \in \Theta(g)$: "f is asymptotically sandwiched between constant multiples of g"

Symmetry of Θ

Lemma

The following statements are equivalent:

1. $f \in \Theta(g)$
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Proof. Suppose that $f \in \Theta(g)$. To show that $g \in \Theta(f)$, we need to prove that there are positive constants C_1, C_2, N_0 such that

$$0 \leq C_1 f(n) \leq g(n) \leq C_2 f(n), \text{for all } n \geq N_0 \ . \qquad (1)$$

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Since $f \in \Theta(g)$, there are positive constants c_1, c_2, n_0 s.t.

$$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n), \text{for all } n \geq n_0. \qquad (2)$$

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Setting $C_1 = \frac{1}{c_2}$, $C_2 = \frac{1}{c_1}, N_0 = n_0$, then (1) is equivalent to (2).

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Further Properties of Θ

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- This is not the case in Fast-Peak-Finding
- However, correct to say that the worst-case runtime of an algorithms is $\Theta(f(n))$

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Big Omega-Notation:

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Only makes sense if best-case runtime is in $\Omega(f)$

Using O, Ω, Θ in Equations

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- O, Ω, Θ are often used in equations

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- This allows us to focus on the essential part of an equation
- Not reversible! E.g., $n + 10 = n + O(1)$ but $n + O(1) \neq n + 10 \ldots$