Lower Bound for Sorting

COMS10018 - Algorithms

Dr Christian Konrad

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- Count number of 0s n_0
- Write n_0 0s followed by $n - n_0$ 1s
- Both operations take time $O(n)$

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Lower Bound for Comparison-based Sorting

- We will prove that every comparison-based sorting algorithm requires $\Omega(n \log n)$ comparisons
- This implies that $O(n \log n)$ is an optimal runtime for comparison-based sorting

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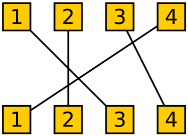

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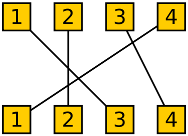

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- A reordering of $[n]$

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Rephrasing our Task:

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Rephrasing our Task: Find permutation $\pi \in \Pi$ such that:

$$A[\pi^{-1}(1)] < A[\pi^{-1}(2)] < \dots < A[\pi^{-1}(n-1)] < A[\pi^{-1}(n)]$$

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Proof. Let the three elements be a, b, c. Suppose that the first query is $a < b$ and suppose that the answer is yes.

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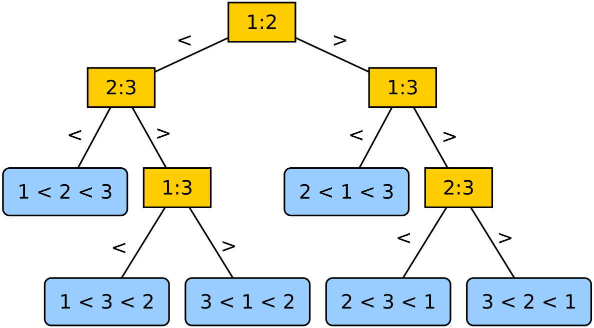
Every Guessing Strategy (and Sorting Algorithm) is a Decision-tree



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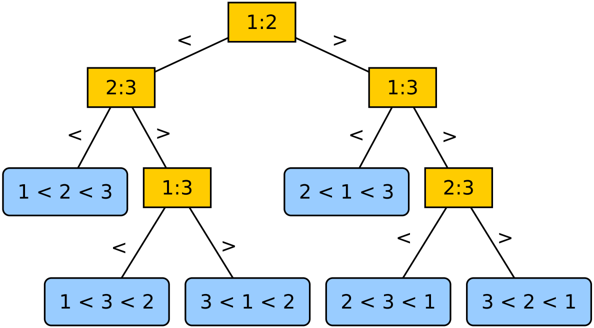

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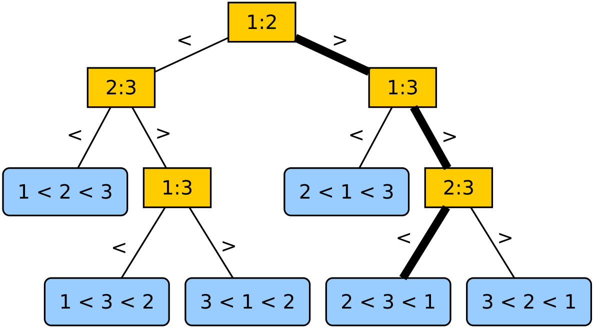

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- An execution is a root-to-leaf path

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$\square$

Stirling's approximation: $n! \geq \left(\frac{n}{e}\right)^n$