

Why Constants Matter Less

COMS10018 - Algorithms

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Runtime of an Algorithm

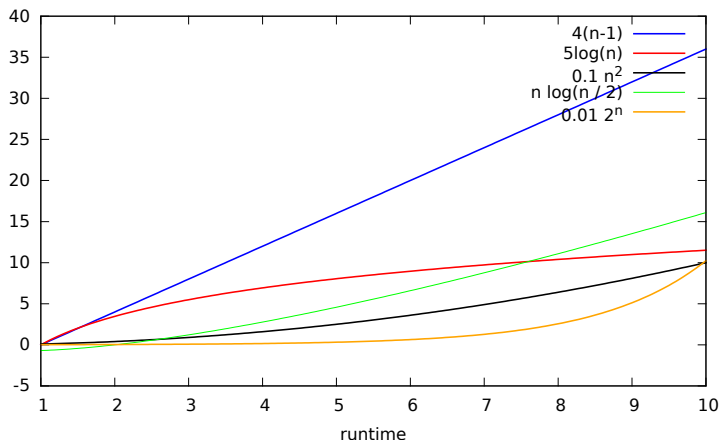
- Function $f : \mathbb{N} \rightarrow \mathbb{N}$ that maps the input length $n \in \mathbb{N}$ to the number of *simple/unit/elementary* operations (worst case, best case, average case, runtime on a specific input, ...)
- The number of array accesses in PEAK FINDING represents the number of unit operations very well

Which runtime is better?

- $4(n - 1)$ (simple peak finding algorithm)
- $5 \log n$ (fast peak finding algorithm)
- $0.1n^2$
- $n \log(0.5n)$
- $0.01 \cdot 2^n$

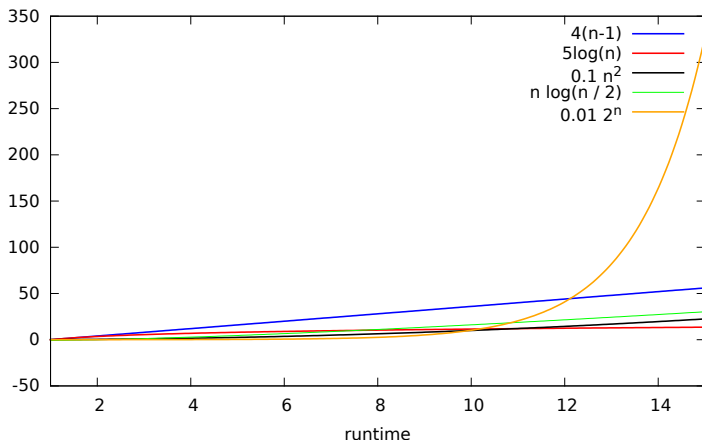
Answer: It depends... But there is a favourite

Runtime Comparisons



$$0.1n^2 \leq 0.01 \cdot 2^n \leq 5 \log n \leq n \log(n/2) \leq 4(n-1) \\ (n = 10)$$

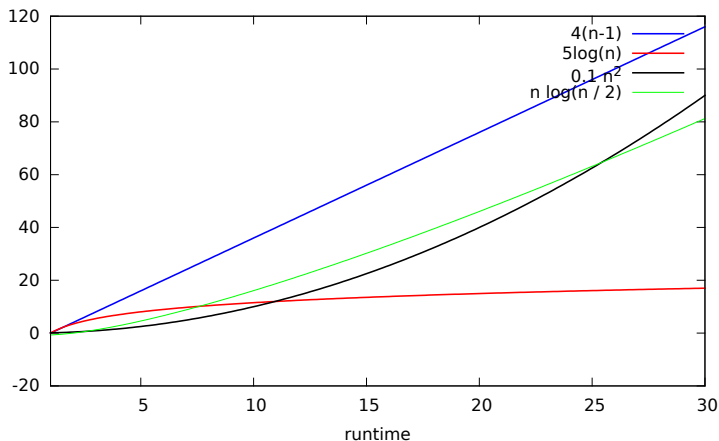
Runtime Comparisons



$$5 \log n \leq 0.1 n^2 \leq n \log(n/2) \leq 4(n-1) \leq 0.01 \cdot 2^n$$

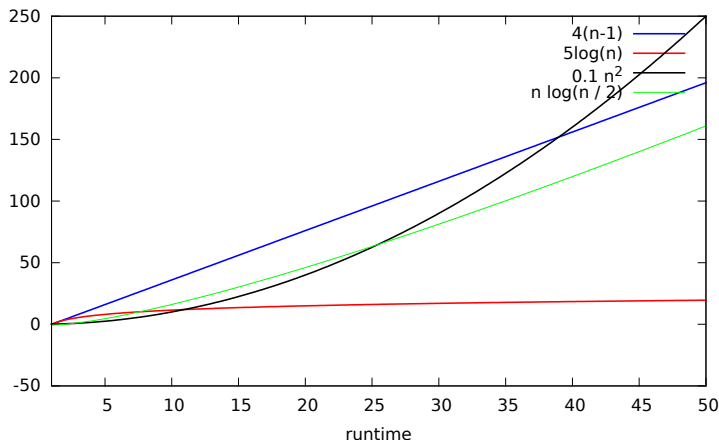
$(n = 15)$

Runtime Comparisons



$$5 \log n \leq n \log(n/2) \leq 0.1n^2 \leq 4(n-1) \\ (n = 30)$$

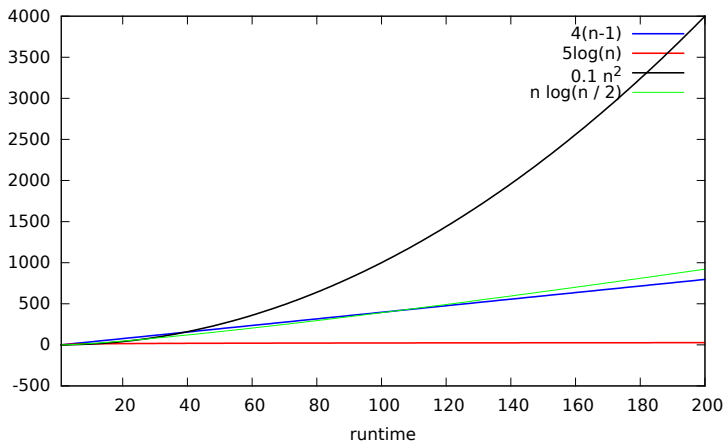
Runtime Comparisons



$$5 \log n \leq n \log(n/2) \leq 4(n-1) \leq 0.1n^2$$

$(n = 50)$

Runtime Comparisons



$$5 \log n \leq 4(n-1) \leq n \log(n/2) \leq 0.1n^2$$

$(n = 200)$

Order Functions Disregarding Constants

Aim: We would like to sort algorithms according to their runtime

Is algorithm A faster than algorithm B ?

Asymptotic Complexity

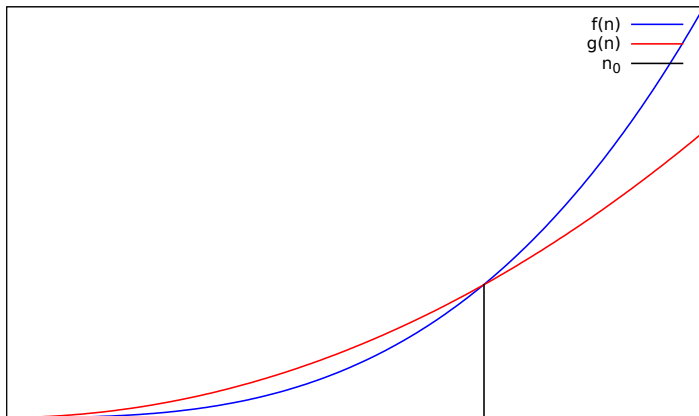
- For large enough n , constants seem to matter less
- For small values of n , most algorithms are fast anyway
(Attention: this is often but not always true!)

Solution: Consider asymptotic behavior of functions

A function $f(n)$ grows *asymptotically at least as fast as* a function $g(n)$ if there exists an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ it holds:

$$f(n) \geq g(n) .$$

Example: f grows at least as fast as g



Example 1

Example: $f(n) = \frac{1}{2}n^2$, $g(n) = 3n$

Then $f(n)$ grows asymptotically at least as fast as $g(n)$.

Proof: Find values of n for which the following holds:

$$\begin{aligned}\frac{1}{2}n^2 &\geq 3n \Rightarrow \\ n &\geq 6.\end{aligned}$$

Thus, we can choose any $n_0 \geq 6$.



Example 2

Example: $f(n) = 2n^3$, $g(n) = \frac{1}{2} \cdot 2^n$

Then $g(n)$ grows asymptotically at least as fast as $f(n)$.

Proof: Find values of n for which the following holds:

$$\begin{aligned}\frac{1}{2} \cdot 2^n &\geq 2n^3 \\ 2^{n-1} &\geq 2^{3 \log n + 1} \quad (\text{using } n = 2^{\log n}) \\ n - 1 &\geq 3 \log n + 1 \\ n &\geq 3 \log n + 2\end{aligned}$$

This holds for every $n \geq 16$ (which follows from the *racetrack principle*). Thus, we chose any $n_0 \geq 16$. □

The Racetrack Principle

Racetrack Principle: Let f, g be functions, k an integer and suppose that the following holds:

- ① $f(k) \geq g(k)$ and
- ② $f'(n) \geq g'(n)$ for every $n \geq k$.

Then for every $n \geq k$, it holds that $f(n) \geq g(n)$.

Example: $n \geq 3 \log n + 2$ holds for every $n \geq 16$

- $n \geq 3 \log n + 2$ holds for $n = 16$
- We have: $(n)' = 1$ and $(3 \log n + 2)' = \frac{3}{n \ln 2} < \frac{1}{2}$ for every $n \geq 16$. The result follows.

Order Functions by Asymptotic Growth

If \leq means *grows asymptotically at least as fast as* then we get:

$$5 \log n \leq 4(n-1) \leq n \log(n/2) \leq 0.1n^2 \leq 0.01 \cdot 2^n$$