Why Constants Matter Less

COMS10018 - Algorithms

Dr Christian Konrad

Runtime of Algorithms

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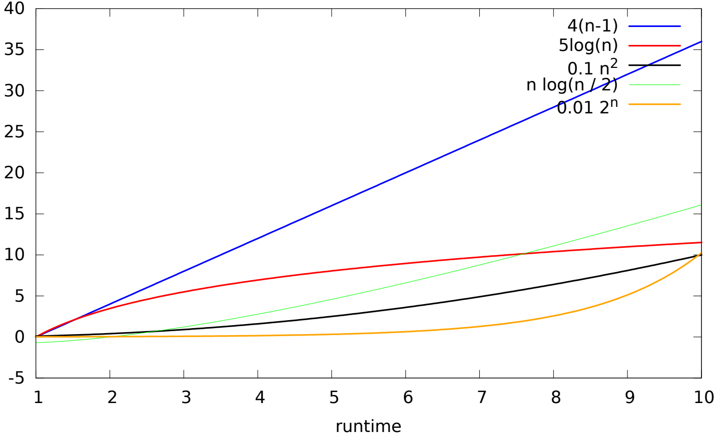
Answer: It depends... But there is a favourite

Runtime Comparisons



$$0.1n^2 \;\; \leq 0.01 \cdot 2^n \leq 5 \log n \leq n \log(n/2) \leq 4(n-1)$$
$$(n = 10)$$

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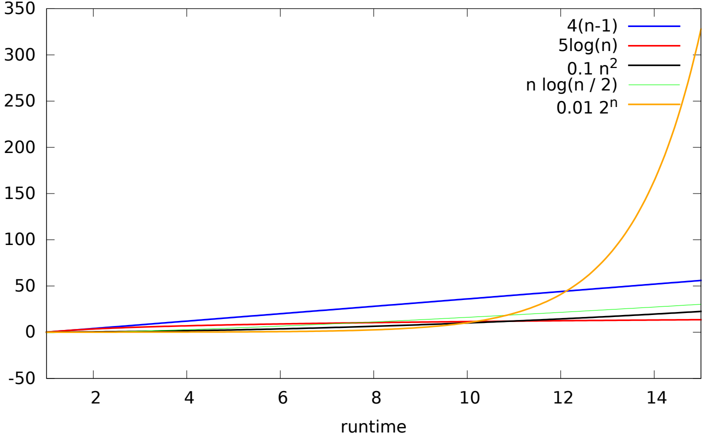

$$5 \log n \leq 0.1n^2 \leq n \log(n/2) \leq 4(n-1) \leq 0.01 \cdot 2^n$$
$$(n = 15)$$

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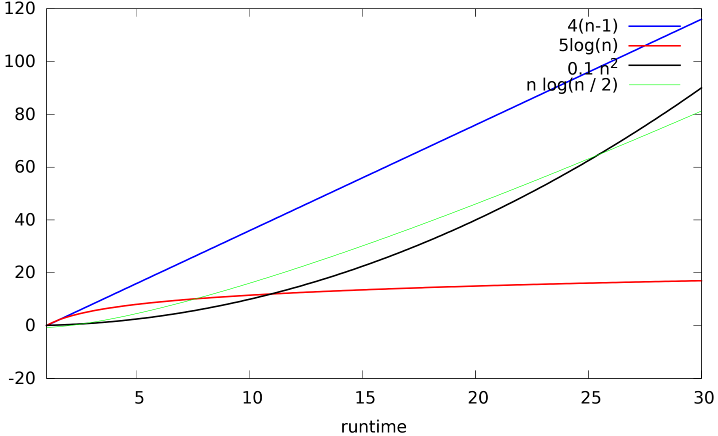

$$5 \log n \le n \log(n/2) \le 0.1 n^2 \le 4(n-1)$$
$$(n = 30)$$

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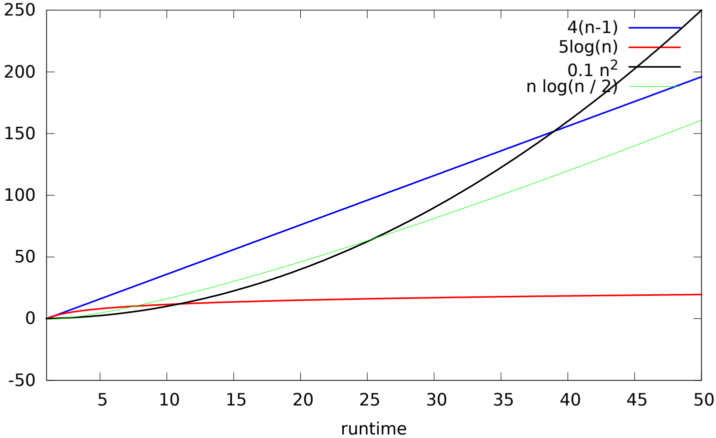

$$\color{red}{5 \log n} \leq \color{green}{n \log(n/2)} \leq \color{blue}{4(n-1)} \leq 0.1 n^2$$
$$(n = 50)$$

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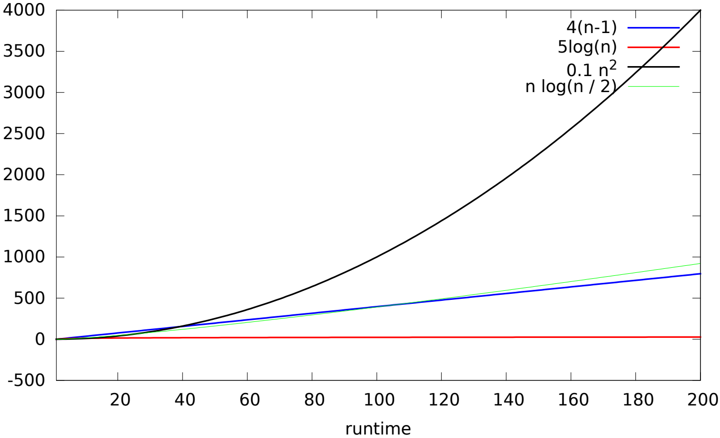

$$5 \log n \leq 4(n-1) \leq n \log(n/2) \leq 0.1n^2$$
$$(n = 200)$$

Order Functions Disregarding Constants

Aim: We would like to sort algorithms according to their runtime

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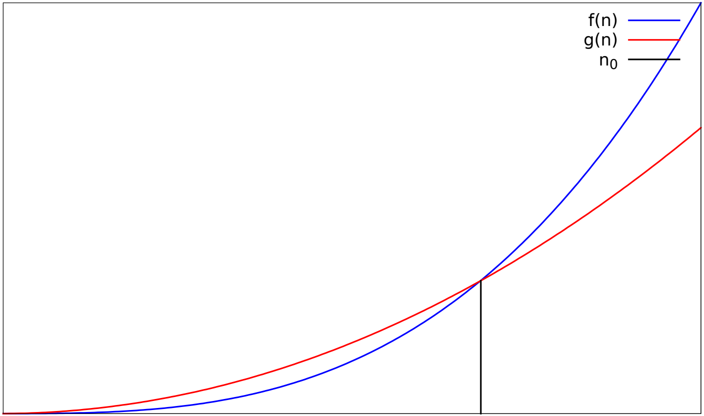
- For small values of n, most algorithms are fast anyway (Attention: this is often but not always true!)

Solution: Consider asymptotic behavior of functions

A function $f(n)$ grows *asymptotically at least as fast as* a function $g(n)$ if there exists an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ it holds:

$$f(n) \geq g(n) \ .$$

Example: f grows at least as fast as g



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Proof: Find values of n for which the following holds:

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Thus, we can chose any $n_0 \geq 6$. $\square$

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$$\begin{aligned} \frac{1}{2} \cdot 2^n &\geq 2n^3 \\ 2^{n-1} &\geq 2^{3 \log n + 1} \quad (\text{using } n = 2^{\log n}) \end{aligned}$$

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This holds for every $n \geq 16$ (which follows from the *racetrack principle*). Thus, we chose any $n_0 \geq 16$. □

The Racetrack Principle

Racetrack Principle: Let f, g be functions, k an integer and suppose that the following holds:

1. $f(k) \geq g(k)$ and
2. $f'(n) \geq g'(n)$ for every $n \geq k$.

Then for every $n \geq k$, it holds that $f(n) \geq g(n)$.

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- $n \geq 3 \log n + 2$ holds for $n = 16$
- We have: $(n)' = 1$ and $(3 \log n + 2)' = \frac{3}{n \ln 2} < \frac{1}{2}$ for every $n \geq 16$. The result follows.

Order Functions by Asymptotic Growth

If $\leq$ means *grows asymptotically at least as fast as* then we get:

$$5\log n \leq 4(n-1) \leq n\log(n/2) \leq 0.1n^2 \leq 0.01 \cdot 2^n$$