Linear and Binary Search

COMS10018 - Algorithms

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Runtime of Algorithms

Consider an algorithm $\mathcal{A}$ for a specific problem $\mathcal{P}$

Set of Potential Inputs

- Let $S(n)$ be the set of all potential inputs of length n for $\mathcal{P}$
- For $X \in S(n)$, let $T(X)$ be the runtime of $\mathcal{A}$ on input X

Worst-case Runtime: $\max_{X \in S(n)} T(X)$

Best-case Runtime: $\min_{X \in S(n)} T(X)$

Average-case Runtime: $\frac{1}{|S(n)|} \sum_{X \in S(n)} T(X)$

Linear Search

Linear Search:

- **Input:** Array A of n integers from range $\{0, 1, 2, \ldots, k-1\}$, for some integer k, integer $t \in \{0, 1, 2, \ldots, k-1\}$
- **Output:** 1, if A contains t, 0 otherwise

Worst-case Runtime: $\Theta(n)$
E.g. on any input with $A[i] \neq t$ for every i

Best-case Runtime: $O(1)$
On any input with $A[0] = t$

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Require: Array A, integer t
  for i = 0, ..., n − 1 do
    if A[i] = t then
      return 1
  return 0
```

Average-case Runtime: (over all possible inputs of length n)

Average-case Analysis of Linear Search

Possible Inputs of Length n

$$\begin{aligned} S(n) &:= \{\text{arrays } A \text{ of length } n \text{ with } A[i] \in \{0, 1, 2, \dots, k-1\}, \\ &\quad \text{for every } 0 \le i \le n-1\} \\ |S(n)| &= k^n \ . \end{aligned}$$

Auxiliary Function: For $A \in S(n), t \in \{0, 1, \dots, k-1\}$:

$$\textsc{Left}(A, t) = \min\{i \ : \ A[i] = t\} \ .$$

If no such position exists then $\textsc{Left}(A, t) = n$.

Examples:

- $\textsc{Left}(23192, 9) = 3$
- $\textsc{Left}(0000, 1) = 4$

$\rightarrow$ Linear search loop executed $\textsc{Left}(X, t) + 1$ times

Average-case Analysis of Linear Search (continued)

Average-case Runtime for $k = 2$: (binary strings)

We compute average number of steps the loop is executed $(t = 1)$

$$\text{AVG} = \frac{1}{|S(n)|} \sum_{A \in S(n)} \textsc{Left}(A, 1) + 1$$

$$= 2^{-n} \left(\left(\sum_{i=0}^{n-1} |\{A : \textsc{Left}(A, 1) = i\}| \cdot (i + 1) \right) + (n + 1) \right) .$$

$$\underbrace{0\ 0\ 0\ 0\ \ldots 0}_{i \text{ times}}\ 1\ \underbrace{X\ X\ X\ \ldots\ X}_{n-i-1 \text{ times}}$$

$$= 2^{-n} \left(\left(\sum_{i=0}^{n-1} 2^{n-1-i} \cdot (i + 1) \right) + (n + 1) \right)$$

$\rightarrow$ **AVG-case runtime is $O(1)$**

$$= \left(\sum_{i=0}^{n-1} \frac{i+1}{2^{i+1}} \right) + (n + 1)2^{-n} \leq 2 + 1 = 3 = O(1) .$$

(Trick for Bounding Sums)

How to bound $\sum_{i=0}^{n} \frac{i}{2^i}$:

$$S_n := \sum_{i=0}^{n} \frac{i}{2^i} \ .$$

Trick: Consider $\frac{1}{2} S_n$

$$\begin{aligned} S_n &= \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots + \frac{n}{2^n} \\ \frac{1}{2} S_n &= \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \cdots + \frac{n}{2^{n+1}} \\ S_n - \frac{1}{2} S_n &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} - \frac{n}{2^{n+1}} \\ &= \left(\sum_{i=1}^{n} \frac{1}{2^i} \right) - \frac{n}{2^{n+1}} \leq \left(\sum_{i=1}^{n} \frac{1}{2^i} \right) \leq 1 \ . \end{aligned}$$

$\rightarrow S_n \leq 2$

Binary Search

Binary Search:

- **Input:** A sorted array A of integers, an integer t
- **Output:** -1 if A does not contain t, otherwise a position i such that $A[i] = t$

Require: Sorted array A of length n, integer t
  if $|A| \leq 2$ **then**
    Check $A[0]$ and $A[1]$ and **return** answer
  if $A[\lfloor n/2 \rfloor] = t$ **then**
    return $\lfloor n/2 \rfloor$
  else if $A[\lfloor n/2 \rfloor] > t$ **then**
    return $\textsc{Binary-Search}(A[0, \ldots, \lfloor n/2 \rfloor - 1])$
  else
    return $\lfloor n/2 \rfloor + 1 + \textsc{Binary-Search}(A[\lfloor n/2 \rfloor + 1, n-1])$

Algorithm $\textsc{Binary-Search}$

Worst-case Analysis of Binary Search

Worst-case Analysis

- Without recursive calls, we spend $O(1)$ time in the function
- Worst-case runtime = $\underbrace{\text{"maximum \# of recursive calls"}}_{r} \cdot O(1)$
- Observe that in iteration i the size of the array is at most half the size than in iteration $i-1$
- We stop as soon as the size of the array is at most two
- Hence, we obtain the necessary and sufficient condition:

$$\frac{n}{2^r} \le 2$$

Solving $\frac{n}{2^r} \le 2$ yields $r \ge \log n - 1$. Hence, $r = \lceil \log n - 1 \rceil \le \log n$ iterations are enough.

Worst-case runtime of Binary Search: $O(\log n)$